

SOLVING $\bar{\partial}$ WITH PRESCRIBED SUPPORT ON HARTOGS TRIANGLES IN \mathbb{C}^2 AND $\mathbb{C}\mathbb{P}^2$

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ABSTRACT. In this paper, we consider the problem of solving the Cauchy–Riemann equation with prescribed support in a domain of a complex manifold for forms or currents. We are especially interested in the case when the domain is a Hartogs triangle in \mathbb{C}^2 or $\mathbb{C}\mathbb{P}^2$. In particular, we show that the strong L^2 Dolbeault cohomology group on the Hartogs triangle in $\mathbb{C}\mathbb{P}^2$ is infinitely dimensional.

In this paper, we consider the problem of solving the Cauchy–Riemann equation with prescribed support. More precisely, let X be a complex manifold of complex dimension n , and let $\Omega \subset X$ be a subdomain of X . We ask the following questions:

Let T be a $\bar{\partial}$ -closed $(r, 1)$ -current, $0 \leq r \leq n$, on X with support contained in $\bar{\Omega}$. Does there exist an $(r, 0)$ -current on X , with support contained in $\bar{\Omega}$, such that $\bar{\partial}S = T$?

If moreover $T = f$ is a smooth form or a C^k -form or an L^p_{loc} -form, can we find g with support contained in $\bar{\Omega}$ and with the same regularity as f such that $\bar{\partial}g = f$?

This leads us to introduce the Dolbeault cohomology groups with prescribed support in $\bar{\Omega}$. Let us denote by $H_{\bar{\Omega}, \infty}^{r,1}(X)$ the quotient space

$$\{f \in C_{r,1}^\infty(X) \mid \bar{\partial}f = 0, \text{ supp } f \subset \bar{\Omega}\} / \bar{\partial}\{f \in C_{r,0}^\infty(X) \mid \text{ supp } f \subset \bar{\Omega}\}.$$

In the same way, we define $H_{\bar{\Omega}, C^k}^{r,1}(X)$, $H_{\bar{\Omega}, L^p_{\text{loc}}}^{r,1}(X)$, and $H_{\bar{\Omega}, \text{cur}}^{r,1}(X)$ for C^k , L^p_{loc} , and the current category.

The cohomology groups $H_{\bar{\Omega}, \infty}^{r,1}(X)$, $H_{\bar{\Omega}, C^k}^{r,1}(X)$, $H_{\bar{\Omega}, L^p_{\text{loc}}}^{r,1}(X)$, and $H_{\bar{\Omega}, \text{cur}}^{r,1}(X)$ describe the obstruction to solve the Cauchy–Riemann equation with prescribed support in $\bar{\Omega}$, respectively, in the smooth or C^k or L^p_{loc} or current category. Their vanishing is equivalent to the solvability of the Cauchy–Riemann equation with prescribed support in $\bar{\Omega}$ in the corresponding category (see [11, section 2], [10]).

Notice that the kernel of $\bar{\partial}$ is always closed by definition. The topology for the quotient space is Hausdorff if and only if the range of $\bar{\partial}$ is closed. If the cohomology group is finite dimensional, then it is trivially Hausdorff since it is isomorphic to R^N for some N . When these groups are infinite dimensional, the Hausdorff property for the quotient topology is equivalent to the closedness of the denominator, which means that the $\bar{\partial}$ operator has closed range (see [18, Proposition 4.5]).

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Note that, if Ω is a relatively compact domain with Lipschitz boundary, by the Serre duality (see [14] or [11]), the properties of the groups $H_{\Omega, \infty}^{r,1}(X)$, $H_{\Omega, L_{\text{loc}}^p}^{r,1}(X)$, and $H_{\Omega, \text{cur}}^{r,1}(X)$ are directly related to the properties of the Dolbeault cohomology groups $\bar{H}^{n-r, n-1}(\Omega)$, $H_{L_{\text{loc}}^{p'}}^{n-r, n-1}(\Omega)$, with $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $H_{\infty}^{n-r, n-1}(\bar{\Omega})$ of Dolbeault cohomology for extendable currents and $L^{p'}$ -forms, and of smooth forms up to the boundary.

If Φ is a family of supports in the complex manifold X , for example, the family, usually denoted by c , of all compact subsets of X , we can consider the Dolbeault cohomology with support in Φ . The group $H_{\Phi, \infty}^{r,q}(X)$ is the quotient of the space of $\bar{\partial}$ -closed, smooth (r, q) -forms on X with support in the family Φ by the range by $\bar{\partial}$ of the space of smooth $(r, q-1)$ -forms on X with support in the family Φ . Similarly, we can define the groups $H_{\Phi, \mathcal{C}^k}^{r,q}(X)$, $H_{\Phi, L_{\text{loc}}^p}^{r,q}(X)$, and $H_{\Phi, \text{cur}}^{r,q}(X)$. It follows from [7, Corollary 2.15], [10, Proposition 1.2] that the Dolbeault isomorphism holds for the Dolbeault cohomology with support condition. This means that all these groups are isomorphic, and we denote them by $H_{\Phi}^{p,q}(X)$. In this paper, we will show that such a Dolbeault isomorphism no longer holds if we change the condition supported in a family of sets in X to prescribed support. For Dolbeault cohomology groups with prescribed support, the following proposition is proved in Proposition 2.1.

Proposition 0.1. *Let X be a complex manifold, and let $\Omega \subset X$ be a domain in X . The natural morphisms from $H_{\Omega, \infty}^{0,1}(X)$ (resp., $H_{\Omega, \mathcal{C}^k}^{0,1}(X)$, $k \geq 0$, $H_{\Omega, L_{\text{loc}}^p}^{0,1}(X)$, $1 \leq p \leq +\infty$) into $H_{\Omega, \text{cur}}^{0,1}(X)$ are injective. In particular, if $H_{\Omega, \text{cur}}^{0,1}(X) = 0$, then $H_{\Omega, \infty}^{0,1}(X) = 0$, $H_{\Omega, \mathcal{C}^k}^{0,1}(X) = 0$, $k \geq 0$, and $H_{\Omega, L_{\text{loc}}^p}^{0,1}(X) = 0$.*

When Ω is a Hartogs triangle type set in \mathbb{C}^2 or $\mathbb{C}\mathbb{P}^2$, we show that the Dolbeault isomorphisms fail to hold for the cohomology with prescribed support. When Ω is an unbounded Hartogs triangle in \mathbb{C}^2 , we get (see Corollary 3.3 and Theorems 3.5, 3.7, and 3.8)

Theorem 0.2. *If $X = \mathbb{C}^2$ and $\Omega = \mathbb{H}^- = \{(z, w) \in \mathbb{C}^2 \mid |z| > |w|\}$, then $H_{\Omega, \infty}^{0,1}(X) = 0$, but $H_{\Omega, \mathcal{C}^k}^{0,1}(X)$, $k \geq 0$, $H_{\Omega, \text{cur}}^{0,1}(X)$, and $H_{\Omega, L_{\text{loc}}^2}^{0,1}(X)$ are infinite dimensional.*

In the case when Ω is a Hartogs triangle in $\mathbb{C}\mathbb{P}^2$, we prove (see Corollaries 4.3 and 4.10 and Theorem 4.5) the following.

Theorem 0.3. *If $X = \mathbb{C}\mathbb{P}^2$ and $\Omega = \mathbb{H}^- = \{[z_0, z_1, z_2] \in \mathbb{C}\mathbb{P}^2 \mid |z_1| > |z_2|\}$, then $H_{\Omega, \infty}^{0,1}(X) = 0$ and $H_{\Omega, \mathcal{C}^k}^{0,1}(X) = 0$, $k \geq 0$, but $H_{\Omega, \text{cur}}^{0,1}(X)$ and $H_{\Omega, L^2}^{0,1}(X)$ are infinite dimensional and Hausdorff.*

The nonvanishing of $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$ is especially interesting since it is in sharp contrast to the case of solving $\bar{\partial}$ with compact support for a bounded Hartogs triangle in \mathbb{C}^2 (see Remark 7). The infinite dimensionality of $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$ gives the following result (see Theorem 4.11):

Let $\bar{\partial}_s$ be the strong L^2 closure of $\bar{\partial}$ on smooth forms up to the boundary in the graph norm. Let $H_{\bar{\partial}_s, L^2}^{2,1}(\mathbb{H}^-)$ be the quotient of the kernel of $\bar{\partial}_s$ over the range of $\bar{\partial}_s$, i.e., the Dolbeault cohomology with respect to the operator $\bar{\partial}_s$.

Theorem 0.4. *Let \mathbb{H}^- be the same as in Theorem 0.3. The space $H_{\bar{\partial}_s, L^2}^{2,1}(\mathbb{H}^-)$ is infinite dimensional.*

It is not known whether $\bar{\partial}_s$ agrees with the weak L^2 extension or if the range of $\bar{\partial}_s$ is closed. If the domain Ω is bounded and Lipschitz, then the weak and strong closure are the same from the Friedrichs lemma. The Hartogs triangle is a candidate in which the weak and strong closure of $\bar{\partial}$ might not be the same.

The vanishing of the Dolbeault cohomology groups with prescribed support in $\bar{\Omega}$ in bidegree $(0, 1)$ is directly related to the extension of holomorphic functions defined on the complement of Ω . This implies the following result (see Corollary 2.6 and Proposition 2.7).

Proposition 0.5. *Let X be a complex manifold, and let $\Omega \subset X$ be a domain in X . Assume that $H_{\Omega, \infty}^{0,1}(X) = 0$; then $X \setminus \Omega$ is connected. If moreover X is not compact, $H_c^{0,1}(X) = 0$, and Ω is relatively compact, then $H_{\Omega, \infty}^{0,1}(X) = 0$ if and only if $X \setminus \Omega$ is connected.*

We also prove (see Theorem 2.14) some characterization of pseudoconvexity for domains in \mathbb{C}^2 in terms of Dolbeault cohomology with the prescribed support.

Theorem 0.6. *Let D be a bounded domain in \mathbb{C}^2 with Lipschitz boundary. Then the following assertions are equivalent:*

- (i) D is a pseudoconvex domain.
- (ii) $H_{D, \infty}^{0,1}(\mathbb{C}^2) = 0$ and $H_{D, \infty}^{0,2}(\mathbb{C}^2)$ is Hausdorff.

The plan of this paper is as follows: In section 1, we recall some basic properties of the support and the uniqueness of the solution for $\bar{\partial}$. In section 2, we discuss solving $\bar{\partial}$ with prescribed support and its relations with the holomorphic extension of functions in various function spaces. In section 3, we study the nonvanishing of Dolbeault cohomology with prescribed support on the unbounded Hartogs triangle in \mathbb{C}^2 . We analyze the Hartogs triangles in $\mathbb{C}\mathbb{P}^2$ in section 4. Theorems 0.2 and 0.3 provide interesting examples which give the nonvanishing for the Dolbeault cohomology groups. This is in sharp contrast with the well-known results of solving $\bar{\partial}$ for $(0,1)$ -forms with prescribed support for a bounded domain in \mathbb{C}^n . We prove the results on the $\bar{\partial}_s$ operator for the domain \mathbb{H}^- in $\mathbb{C}\mathbb{P}^2$ using L^2 Serre duality. This gives us some insight about the intriguing problem on weak and strong extensions of the $\bar{\partial}$ operator in the L^2 sense, when the domain is not Lipschitz. The unbounded Hartogs domain in \mathbb{C}^2 or Hartogs domains in $\mathbb{C}\mathbb{P}^2$ provide us with new unexpected phenomena. Many open questions and remarks are given at the end of the paper.

1. PROPERTIES OF THE SUPPORT AND UNIQUENESS OF THE SOLUTION

Let X be a complex manifold of complex dimension n , and let T be a $\bar{\partial}$ -exact $(0,1)$ -current on X . We will describe some relations between the support of the current T and the support of the solution S of the Cauchy–Riemann equation $\bar{\partial}S = T$.

Proposition 1.1. *Let X be a complex manifold of complex dimension n , and let T be a $\bar{\partial}$ -exact $(0,1)$ -current on X . If Ω^c denotes a connected component of $X \setminus \text{supp } T$ and if S is a distribution on X such that $\bar{\partial}S = T$, then either $\text{supp } S \cap \Omega^c = \emptyset$ or $\Omega^c \subset \text{supp } S$.*

Proof. Note that, since $\bar{\partial}S = T$, S is a holomorphic function on $X \setminus \text{supp } T$ and in particular on the connected set Ω^c . Assume that the support of S does not contain Ω^c . Then S vanishes on an open subset of Ω^c and by analytic continuation S vanishes on Ω^c , which means that $\text{supp } S \cap \Omega^c = \emptyset$. \square

Corollary 1.2. *Let X be a complex manifold of complex dimension n , and let T be a $\bar{\partial}$ -exact $(0, 1)$ -current on X . Assume that $X \setminus \text{supp } T$ is connected. Then if S is a distribution on X such that $\bar{\partial}S = T$, then either $\text{supp } S = \text{supp } T$ or $\text{supp } S = X$.*

Proof. The support of T is always contained in the support of S . If $\text{supp } S \neq X$, then the other inclusion holds by Proposition 1.1 since $X \setminus \text{supp } T$ is connected. \square

Note that the difference between two solutions of the equation $\bar{\partial}S = T$ is a holomorphic function on X . Then analytic continuation implies the following uniqueness result.

Proposition 1.3. *Assume that the complex manifold X is connected. Let T be a $\bar{\partial}$ -exact $(0, 1)$ -current on X such that $X \setminus \text{supp } T \neq \emptyset$, and let S and U be two distributions such that*

$$\bar{\partial}S = \bar{\partial}U = T$$

and there exists a connected component Ω^c of $X \setminus \text{supp } T$ such that

$$\text{supp } S \cap \Omega^c = \text{supp } U \cap \Omega^c = \emptyset.$$

Then $S = U$.

In particular, the equation $\bar{\partial}S = T$ admits at most one solution S such that $\text{supp } S = \text{supp } T$.

Remark 1. The equation $\bar{\partial}S = T$ may have no solution S with $\text{supp } S = \text{supp } T$. Consider, for example, a relatively compact domain D with C^∞ -smooth boundary in a complex manifold X and a function $F \in C^\infty(\bar{D})$ which is holomorphic in D . Denote by f the restriction of F to the boundary of D , and set $S = F\chi_D$, where χ_D is the characteristic function of the domain D . Then, by the Stokes formula, $\bar{\partial}S = f[\partial D]^{0,1}$, where $[\partial D]^{0,1}$ is the part of bidegree $(0, 1)$ of the integration current over the boundary of D . Clearly the support of $T = f[\partial D]^{0,1}$ is the boundary of D , but, by Proposition 1.3, S is the unique solution of $\bar{\partial}S = T$ whose support is contained in \bar{D} . So there is no solution whose support is equal to the support of T .

Let us end this section by considering the regularity of the solutions.

Proposition 1.4. *Let X be a complex manifold, and let f be a $(0, 1)$ -form with coefficients in $C^k(X)$, $0 \leq k \leq +\infty$ (resp., $L_{\text{loc}}^p(X)$, $1 \leq p \leq +\infty$), which is $\bar{\partial}$ -exact in the sense of currents. Then any solution g of the equation $\bar{\partial}g = f$ is in $C^k(X)$, $0 \leq k \leq +\infty$ (resp., $L_{\text{loc}}^p(X)$, $1 \leq p \leq +\infty$).*

Proof. By the regularity of the Cauchy–Riemann operator (injectivity of the Dolbeault isomorphism [7], [10]), if f has coefficients in $C^k(X)$, $0 \leq k \leq +\infty$ (resp., $L_{\text{loc}}^p(X)$, $1 \leq p \leq +\infty$), then, since f is $\bar{\partial}$ -exact in the sense of currents, the equation $\bar{\partial}S = f$ has a solution in $C^k(X)$, $0 \leq k \leq +\infty$ (resp., $L_{\text{loc}}^p(X)$, $1 \leq p \leq +\infty$). With the difference between two solutions of the equation $\bar{\partial}S = f$ being a holomorphic function on X , all of the solutions have the same regularity. \square

Associating Propositions 1.3 and 1.4, we get the following corollary.

Corollary 1.5. *Assume that the complex manifold X is connected. If f is a $(0, 1)$ -form such that $X \setminus \text{supp } f \neq \emptyset$, then the equation $\bar{\partial}g = f$ has at most one unique solution such that $\text{supp } g = \text{supp } f$, and this solution has the same regularity as f .*

2. SOLVING $\bar{\partial}$ WITH PRESCRIBED SUPPORT

Let X be a connected, complex manifold, and let Ω be a domain such that $\bar{\Omega}$ is strictly contained in X and that the interior of $\bar{\Omega}$ coincides with Ω . We set $\Omega^c = X \setminus \bar{\Omega}$, it is a nonempty open subset of X .

Let us denote by $H_{\bar{\Omega}, \infty}^{0,1}(X)$ (resp., $H_{\bar{\Omega}, \text{cur}}^{0,1}(X)$, $H_{\bar{\Omega}, \mathcal{C}^k}^{0,1}(X)$, $H_{\bar{\Omega}, L_{\text{loc}}^p}^{0,1}(X)$) the Dolbeault cohomology group of bidegree $(0, 1)$ for smooth forms (resp., currents, \mathcal{C}^k -forms, $k \geq 0$, L_{loc}^p -forms, $1 \leq p \leq +\infty$) with support in $\bar{\Omega}$. The vanishing of these groups means that one can solve the $\bar{\partial}$ equation with prescribed support in $\bar{\Omega}$ in the smooth category (resp., the space of currents, the space of \mathcal{C}^k -forms, the space of L_{loc}^p -forms).

Propositions 1.3 and 1.4 and the Dolbeault isomorphism with support conditions [7, Corollary 2.15], [10, Proposition 1.2] lead to the following.

Proposition 2.1. *The natural morphisms from $H_{\bar{\Omega}, \infty}^{0,1}(X)$ (resp., $H_{\bar{\Omega}, \mathcal{C}^k}^{0,1}(X)$, $k \geq 0$, $H_{\bar{\Omega}, L_{\text{loc}}^p}^{0,1}(X)$, $1 \leq p \leq +\infty$) into $H_{\bar{\Omega}, \text{cur}}^{0,1}(X)$ are injective. In particular, if $H_{\bar{\Omega}, \text{cur}}^{0,1}(X) = 0$, then $H_{\bar{\Omega}, \infty}^{0,1}(X) = 0$, $H_{\bar{\Omega}, \mathcal{C}^k}^{0,1}(X) = 0$, and $H_{\bar{\Omega}, L_{\text{loc}}^p}^{0,1}(X) = 0$.*

In the next sections, examples are given proving that there exist domains in \mathbb{C}^2 and $\mathbb{C}P^2$ such that $H_{\bar{\Omega}, \infty}^{0,1}(X) = 0$, but $H_{\bar{\Omega}, \text{cur}}^{0,1}(X) \neq 0$.

We will now consider the link between the vanishing of the group $H_{\bar{\Omega}, \text{cur}}^{0,1}(X)$ and the extension properties of some holomorphic functions in Ω^c .

Proposition 2.2. *Assume that $H_{\bar{\Omega}, \text{cur}}^{0,1}(X) = 0$. Then any holomorphic function on $\Omega^c = X \setminus \bar{\Omega}$ which is the restriction to Ω^c of a distribution on X extends as a holomorphic function to X .*

Proof. Let $f \in \mathcal{O}(\Omega^c)$ and $S_f \in \mathcal{D}'(X)$ be a distribution such that $S_f|_{\Omega^c} = f$. Consider the $(0, 1)$ -current $\bar{\partial}S_f$; it is closed and has support in $\bar{\Omega}$. Since $H_{\bar{\Omega}, \text{cur}}^{0,1}(X) = 0$, there exists $U \in \mathcal{D}'(X)$, with support in $\bar{\Omega}$ such that $\bar{\partial}U = \bar{\partial}S_f$ in X . Set $h = S_f - U$. It is a holomorphic function on X and $h|_{\Omega^c} = S_f|_{\Omega^c} = f$. \square

In the same way, we can prove the following.

Proposition 2.3. *Assume that $H_{\bar{\Omega}, L_{\text{loc}}^p}^{0,1}(X) = 0$, $p \geq 1$. Then any holomorphic function on $\Omega^c = X \setminus \bar{\Omega}$ which is the restriction to Ω^c of a form with coefficients in $W_{\text{loc}}^{1,p}(X)$ extends as a holomorphic function to X .*

Proposition 2.4. *Assume that $H_{\bar{\Omega}, \mathcal{C}^k}^{0,1}(X) = 0$, $k \geq 0$. Then any holomorphic function on $\Omega^c = X \setminus \bar{\Omega}$ which is of class \mathcal{C}^{k+1} on $X \setminus \Omega = \bar{\Omega}^c$ extends as a holomorphic function to X .*

Proposition 2.5. *Assume that $H_{\bar{\Omega}, \infty}^{0,1}(X) = 0$. Then any holomorphic function on $\Omega^c = X \setminus \bar{\Omega}$ which is smooth on $X \setminus \Omega = \bar{\Omega}^c$ extends as a holomorphic function to X .*

Corollary 2.6. *Assume that $H_{\Omega, \infty}^{0,1}(X) = 0$. Then $\Omega^c = X \setminus \overline{\Omega}$ is connected.*

Proof. Assume that Ω^c is not connected. Let f be a holomorphic function which is a constant equal to 1 in one connected component of Ω^c and vanishes identically on all the other ones. By analytic continuation, f cannot be the restriction to Ω^c of a holomorphic function on X , and by Proposition 2.5 we get $H_{\Omega, \infty}^{0,1}(X) \neq 0$. \square

Remark 2. Note that, by Proposition 1.1, $H_{\Omega, \text{cur}}^{0,1}(X) \neq 0$ if and only if there exists at least one $\bar{\partial}$ -exact $(0, 1)$ -current T with support contained in $\overline{\Omega}$ such that the support of each solution of the equation $\bar{\partial}S = T$ contains at least a connected component of Ω^c .

Let us give a partial converse to Corollary 2.6. Let $H_c^{0,1}(X)$ denote the Dolbeault cohomology group for $(0, 1)$ -forms with compact support in X .

Proposition 2.7. *Assume that Ω is relatively compact in a noncompact complex manifold X such that $H_c^{0,1}(X) = 0$. If $\Omega^c = X \setminus \overline{\Omega}$ is connected, then*

$$H_{\Omega, \text{cur}}^{0,1}(X) = H_{\Omega, \infty}^{0,1}(X) = H_{\Omega, C^k}^{0,1}(X) = H_{\Omega, L^p_{\text{loc}}}^{0,1}(X) = 0.$$

Proof. By Proposition 1.4, it suffices to prove that $H_{\Omega, \text{cur}}^{0,1}(X) = 0$. This vanishing result follows directly from Proposition 1.1. More precisely, if T is a $\bar{\partial}$ -closed current on X with support contained in $\overline{\Omega}$, there exists a distribution S with compact support such that $\bar{\partial}S = T$ since $H_c^{0,1}(X) = 0$. Then the support of S cannot contain the connected set Ω^c , otherwise $X = \overline{\Omega} \cup \text{supp } S$ would be compact, and hence $\text{supp } S$ is contained in $\overline{\Omega}$. \square

In particular, if X is a Stein manifold with $\dim_{\mathbb{C}} X \geq 2$ and Ω a relatively compact domain in X , then

$$H_{\Omega, \text{cur}}^{0,1}(X) = H_{\Omega, \infty}^{0,1}(X) = H_{\Omega, C^k}^{0,1}(X) = H_{\Omega, L^p_{\text{loc}}}^{0,1}(X) = 0 \Leftrightarrow \Omega^c \text{ is connected.}$$

An immediate corollary of Propositions 2.7 and 2.2 is the following.

Corollary 2.8. *Let X be a noncompact, connected complex manifold such that $H_c^{0,1}(X) = 0$, and let Ω be a relatively compact, open subset of X with connected complement. Then any holomorphic function on Ω^c extends as a holomorphic function to X .*

Proof. It is sufficient to apply Propositions 2.7 and 2.2 to a neighborhood D of $\overline{\Omega}$ with connected complement and to conclude by analytic continuation. \square

Corollary 2.8 is the classical Hartogs extension phenomenon. Note that all of the previous results remain true if we replace the family of all compact subsets of a noncompact manifold by any family Φ of supports in a manifold X , unlike the family of all closed subsets of X (see, e.g., [14] for the definition of a family of supports).

Proposition 2.9. *Assume that the complex manifold X satisfies $H^{0,1}(X) = 0$. If any holomorphic function on Ω^c which is smooth on $X \setminus \Omega = \overline{\Omega}^c$ extends as a holomorphic function to X , then $H_{\Omega, \infty}^{0,1}(X) = 0$.*

Proof. Let f be a smooth $\bar{\partial}$ -closed form in X with support contained in $\bar{\Omega}$. Since $H^{0,1}(X) = 0$, there exists a function $g \in C^\infty(X)$ such that $\bar{\partial}g = f$. Since the support of f is contained in $\bar{\Omega}$, g is holomorphic in Ω^c , and by the extension property it extends as a holomorphic function \tilde{g} to X . Set $h = g - \tilde{g}$. Then the support of h is contained in $\bar{\Omega}$ and $\bar{\partial}h = f$. \square

Similarly, since $H^{0,1}(X) = H_{C^k}^{0,1}(X) = H_{L_{loc}^p}^{0,1}(X) = H_{cur}^{0,1}(X) = 0$ by the Dolbeault isomorphism, we have the following.

Proposition 2.10. *Assume that the complex manifold X satisfies $H^{0,1}(X) = 0$. If any holomorphic function on Ω^c which is of class C^k , $k \geq 0$ on $X \setminus \Omega = \bar{\Omega}^c$ extends as a holomorphic function to X , then $H_{\bar{\Omega}, C^k}^{0,1}(X) = 0$.*

Proposition 2.11. *Assume that the complex manifold X satisfies $H^{0,1}(X) = 0$. If any holomorphic function on $\Omega^c = X \setminus \bar{\Omega}$ which is the restriction to Ω^c of a function $L_{loc}^p(X)$, $p \geq 1$ extends as a holomorphic function to X , then $H_{\bar{\Omega}, L_{loc}^p}^{0,1}(X) = 0$.*

Proposition 2.12. *Assume that the complex manifold X satisfies $H^{0,1}(X) = 0$. If any holomorphic function on $\Omega^c = X \setminus \bar{\Omega}$ which is the restriction to Ω^c of a distribution on X extends as a holomorphic function to X , then $H_{\bar{\Omega}, cur}^{0,1}(X) = 0$.*

Let us end this section by a characterization of pseudoconvexity in \mathbb{C}^2 by means of the Dolbeault cohomology with prescribed support.

Theorem 2.13. *Let D be a bounded domain in \mathbb{C}^2 with Lipschitz boundary. Then the following assertions are equivalent:*

- (i) D is a pseudoconvex domain.
- (ii) $H_{\bar{D}, \infty}^{0,1}(\mathbb{C}^2) = 0$, and $H_{\bar{D}, \infty}^{0,2}(\mathbb{C}^2)$ is Hausdorff.

Proof. By Serre duality ([3] or [11, Theorem 2.7]) assertion (ii) implies that $\check{H}^{2,q}(D)$ is Hausdorff for all $1 \leq q \leq 2$, and moreover $\check{H}^{2,1}(D) = 0$ as the dual space to $H_{\bar{D}, \infty}^{0,1}(\mathbb{C}^2)$. Let us prove now that the condition $\check{H}^{2,1}(D) = 0$ implies that D is pseudoconvex. We will follow the methods used by Laufer [9] for the usual Dolbeault cohomology and prove by contradiction.

Assume that D is not pseudoconvex. Then there exists a domain \tilde{D} strictly containing D such that any holomorphic function on D extends holomorphically to \tilde{D} . Since interior $(\bar{D}) = D$, after a translation and a rotation, we may assume that $0 \in \tilde{D} \setminus \bar{D}$ and that there exists a point z_0 in the intersection of the plane $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 0\}$ with D which belongs to the same connected component of the intersection of that plane with \tilde{D} .

Let us denote by $B(z_1, z_2)$ the $(0, 1)$ -form defined by

$$B(z_1, z_2) = \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{|z|^4} \wedge dz_1 \wedge dz_2.$$

It is derived from the Bochner–Martinelli kernel in \mathbb{C}^2 and is a $\bar{\partial}$ -closed form on $\mathbb{C}^2 \setminus \{0\}$. Then the L_{loc}^1 -form $\frac{\bar{z}_2}{|z|^2} \wedge dz_1 \wedge dz_2$ defines a distribution in \mathbb{C}^2 which satisfies

$$\bar{\partial}\left(\frac{\bar{z}_2}{|z|^2} dz_1 \wedge dz_2\right) = z_1 B(z_1, z_2) \text{ on } \mathbb{C}^2 \setminus \{0\}.$$

On the other hand, if $\check{H}^{2,1}(D) = 0$, there exists an extendable $(2, 0)$ -current v such that $\bar{\partial}v = B$ on D , and by the regularity of $\bar{\partial}$ in bidegree $(2, 1)$, v is smooth on D since B is smooth on $\mathbb{C}^2 \setminus \{0\}$. Set

$$F = z_1 v - \frac{\bar{z}_2}{|z|^2} \wedge dz_1 \wedge dz_2.$$

Then F is a holomorphic $(2, 0)$ -form on D , so its coefficient F_{12} should extend holomorphically to \tilde{D} , but we have $F_{12}(0, z_2) = \frac{1}{z_2}$ on $D \cap \{z_1 = 0\}$, which is holomorphic and singular at $z_2 = 0$. This gives the contradiction since $0 \in \tilde{D} \setminus D$. This proves that (ii) \Rightarrow (i).

For the converse, first note that if D is a pseudoconvex domain in \mathbb{C}^2 , then $\mathbb{C}^2 \setminus D$ is connected and, by Proposition 2.7, we have $H_{\bar{D}, \infty}^{0,1}(\mathbb{C}^2) = 0$. Then we apply [4, Theorem 5] to get that if D is pseudoconvex with Lipschitz boundary, then $H_{\infty}^{0,1}(\mathbb{C}^2 \setminus D)$ is Hausdorff. Let us prove that if $H_{\infty}^{0,1}(\mathbb{C}^2 \setminus D)$ is Hausdorff, then $H_{\bar{D}, \infty}^{0,2}(\mathbb{C}^2)$ is Hausdorff.

Let f be a $\bar{\partial}$ -closed $(0, 2)$ -form on \mathbb{C}^2 with support contained in \bar{D} such that for any $\bar{\partial}$ -closed $(2, 0)$ -current T on D extendable as a current to \mathbb{C}^2 , we have $\langle T, f \rangle = 0$. Since $H^{0,2}(\mathbb{C}^2) = 0$, there exists a smooth $(0, 1)$ -form g on \mathbb{C}^2 such that $\bar{\partial}g = f$ on \mathbb{C}^2 , in particular $\bar{\partial}g = 0$ on $\mathbb{C}^2 \setminus \bar{D}$.

Let S be any $\bar{\partial}$ -closed $(2, 1)$ -current on \mathbb{C}^2 with compact support in $\mathbb{C}^2 \setminus D$. Then, since $H_c^{2,1}(\mathbb{C}^2) = 0$, there exists a compactly supported $(2, 0)$ -current U on \mathbb{C}^2 such that $\bar{\partial}U = S$ and in particular $\bar{\partial}U = 0$ on D .

Thus

$$\langle S, g \rangle = \langle \bar{\partial}U, g \rangle = \langle U, \bar{\partial}g \rangle = \langle U, f \rangle = 0,$$

by hypothesis on f . Therefore the Hausdorff property of $H_{\infty}^{0,1}(\mathbb{C}^2 \setminus D)$ implies that there exists a smooth function h on $\mathbb{C}^2 \setminus D$ such that $\bar{\partial}h = g$. Let \tilde{h} be a smooth extension of h to \mathbb{C}^2 . Then $u = g - \bar{\partial}\tilde{h}$ is a smooth form with support in \bar{D} and

$$\bar{\partial}u = \bar{\partial}(g - \bar{\partial}\tilde{h}) = \bar{\partial}g = f.$$

This proves that $H_{\bar{D}, \infty}^{0,2}(\mathbb{C}^2)$ is Hausdorff, which proves that (i) \Rightarrow (ii). □

3. THE CASE OF THE UNBOUNDED HARTOGS TRIANGLE IN \mathbb{C}^2

In \mathbb{C}^2 , let us define the domains \mathbb{H}^+ and \mathbb{H}^- by

$$\mathbb{H}^+ = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w|\},$$

$$\mathbb{H}^- = \{(z, w) \in \mathbb{C}^2 \mid |z| > |w|\}.$$

Then $\mathbb{H}^+ \cap \mathbb{H}^- = \emptyset$ and $\overline{\mathbb{H}^+} \cup \overline{\mathbb{H}^-} = \mathbb{C}^2$.

Let us denote by $H_{\overline{\mathbb{H}^-}, \infty}^{0,1}(\mathbb{C}^2)$ (resp., $H_{\overline{\mathbb{H}^-}, \text{cur}}^{0,1}(\mathbb{C}^2)$, $H_{\overline{\mathbb{H}^-}, L^2_{\text{loc}}}^{0,1}(\mathbb{C}^2)$, $H_{\overline{\mathbb{H}^-}, \mathcal{C}^k}^{0,1}(\mathbb{C}^2)$) the Dolbeault cohomology group of bidegree $(0, 1)$ for smooth forms (resp., currents, L^2_{loc} -forms, \mathcal{C}^k -forms) with support in $\overline{\mathbb{H}^-}$.

The vanishing of these groups means that one can solve the $\bar{\partial}$ equation with prescribed support in $\overline{\mathbb{H}^-}$ in the smooth category (resp., the space of currents, the space of L^2 -forms, the space of \mathcal{C}^k -forms).

We can apply Propositions 2.5 and 2.10 for $\Omega = \mathbb{H}^-$ since $H^{0,1}(\mathbb{C}^2) = 0$, and we get the following.

Proposition 3.1. *We have $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2) = 0$ if and only if any holomorphic function on \mathbb{H}^+ which is smooth on $\overline{\mathbb{H}^+}$ extends as a holomorphic function to \mathbb{C}^2 .*

Proposition 3.2. *Any holomorphic function on \mathbb{H}^+ which is smooth on $\overline{\mathbb{H}^+}$ extends as a holomorphic function to \mathbb{C}^2 .*

Proof. Let $f \in C^\infty(\overline{\mathbb{H}^+}) \cap \mathcal{O}(\mathbb{H}^+)$. By Sibony's result [16, page 220], for any $R > 0$, the restriction of f to $\mathbb{H}^+ \cap \Delta(0, R) \times \Delta(0, R)$ extends holomorphically to the bidisc $\Delta(0, R) \times \Delta(0, R)$ and then by analytic continuation f extends holomorphically to \mathbb{C}^2 . \square

Propositions 3.1 and 3.2 immediately lead to the following.

Corollary 3.3. $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2) = 0$.

Let us consider now the case of currents. We can apply Proposition 2.4 to get the following.

Proposition 3.4. *Assume that we have $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2) = 0$, $k \geq 0$. Then any holomorphic function on \mathbb{H}^+ , which is of class \mathcal{C}^{k+1} on $\overline{\mathbb{H}^+}$, extends as a holomorphic function to \mathbb{C}^2 .*

Theorem 3.5. *For any $k \geq 0$, $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2)$ is infinite dimensional.*

Proof. Consider the function h_l defined on \mathbb{H}^+ by $h_l(z, w) = z^l(\frac{z}{w})$, $l \geq 0$. It is of class \mathcal{C}^{k+1} on $\overline{\mathbb{H}^+}$ if $l \geq k + 2$, but it does not extend as a holomorphic function to \mathbb{C}^2 . In fact, if h_l admits a holomorphic extension H_l to \mathbb{C}^2 , then we would have

$$H_l(z, w) = z^l(\frac{z}{w}) \text{ on } \mathbb{C}^2 \setminus \{w = 0\},$$

which is not bounded nearby $\{(z, w) \in \mathbb{C}^2 \mid z \neq 0, w = 0\}$. By Proposition 3.4, we get $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2) \neq 0$.

For $l \geq k + 2$, let \tilde{h}_l be an extension of class \mathcal{C}^{k+1} of h_l to \mathbb{C}^2 . We set $g_l = \bar{\partial}\tilde{h}_l$. Then $\bar{\partial}g_l = 0$ and $\text{supp } g_l \subset \overline{\mathbb{H}^+}$, so the cohomology class $[g_l]$ belongs to $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2)$. We shall prove that the cohomology classes $[g_l]$, $l \geq k + 2$ are linearly independent, and hence $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2)$ is infinite dimensional. For any $N \geq 1$, we set $G_N = \sum_{l=1}^N c_l g_l$ for some complex constants c_l , and we assume that $[G_N] = 0$ in $H_{\mathbb{H}^+}^{0,1}(\mathbb{C}^2)$, which means that there exists a \mathcal{C}^k function u with support contained in $\overline{\mathbb{H}^+}$ such that $\bar{\partial}u = G_N$. Set $H_N = \sum_{l=1}^N c_l h_l$. The function H_N is holomorphic in \mathbb{H}^+ and $\tilde{H}_N = \sum_{l=1}^N c_l \tilde{h}_l$ is an extension of class \mathcal{C}^{k+1} of H_N to \mathbb{C}^2 , which verifies that $\bar{\partial}\tilde{H}_N = G_N$, and therefore $\tilde{H}_N - u$ is a holomorphic extension of H_N to \mathbb{C}^2 . Moreover by analytic continuation,

$$\tilde{H}_N - u = (\frac{z}{w}) \left(\sum_{l=1}^N c_l z^l \right) \text{ on } \mathbb{C}^2 \setminus \{w = 0\},$$

which is not bounded nearby $\{(z, w) \in \mathbb{C}^2 \mid z \neq 0, w = 0\}$ unless $c_1 = \dots = c_N = 0$. \square

Proposition 3.1 still holds if we replace smooth forms by W_{loc}^1 -forms (for $D \subset \mathbb{C}^2$, $W_{\text{loc}}^1(\overline{D})$ is the space of functions which are in $W^1(\overline{D} \cap B(0, R))$ for any $R > 0$) in the following way.

Proposition 3.6. *We have $H_{\mathbb{H}^-, L_{\text{loc}}^2}^{0,1}(\mathbb{C}^2) = 0$ if and only if any function $f \in \mathcal{O}(\mathbb{H}^+) \cap W_{\text{loc}}^1(\overline{\mathbb{H}^+})$, which is the restriction to $\overline{\mathbb{H}^+}$ of a form with coefficients in $W_{\text{loc}}^1(\mathbb{C}^2)$, extends as a holomorphic function to \mathbb{C}^2 .*

Theorem 3.7. *$H_{\mathbb{H}^-, L_{\text{loc}}^2}^{0,1}(\mathbb{C}^2)$ is infinite dimensional.*

Proof. Let us consider the function h_l defined on \mathbb{H}^+ by $h_l(z, w) = z^l(\frac{z}{w})$, $l \geq 3$. It is of class \mathcal{C}^2 on $\overline{\mathbb{H}^+}$, and it is in $W_{\text{loc}}^1(\overline{\mathbb{H}^+})$ and extends as a \mathcal{C}^2 function to \mathbb{C}^2 by the Whitney extension theorem, but it does not extend as a holomorphic function to \mathbb{C}^2 . In fact, if h_l would admit a holomorphic extension H_l to \mathbb{C}^2 , then we would have

$$H_l(z, w) = z^l(\frac{z}{w}) \quad \text{on } \mathbb{C}^2 \setminus \{w = 0\},$$

which is not bounded nearby $\{(z, w) \in \mathbb{C}^2 \mid z \neq 0, w = 0\}$. By Proposition 3.6, we get $H_{\mathbb{H}^-, L_{\text{loc}}^2}^{0,1}(\mathbb{C}^2) \neq 0$.

In the same way as for the \mathcal{C}^k case, we get that in fact $H_{\mathbb{H}^-, L_{\text{loc}}^2}^{0,1}(\mathbb{C}^2)$ is infinite dimensional. □

Theorem 3.8. *$H_{\mathbb{H}^-, \text{cur}}^{0,1}(\mathbb{C}^2)$ is infinite dimensional and Hausdorff.*

Proof. Using Proposition 2.1, it follows from Theorem 3.5 that $H_{\mathbb{H}^-, \text{cur}}^{0,1}(\mathbb{C}^2)$ is infinite dimensional. By the Serre duality, to prove that $H_{\mathbb{H}^-, \text{cur}}^{0,1}(\mathbb{C}^2)$ is Hausdorff, it is sufficient to prove that $H_{\infty}^{2,2}(\overline{\mathbb{H}^-}) = 0$.

Let f be a smooth $(2, 2)$ -form on $\overline{\mathbb{H}^-}$. Then f extends as a smooth $(2, 2)$ -form on \mathbb{C}^2 , called \tilde{f} . Since the top degree Dolbeault cohomology group $H^{2,2}(\mathbb{C}^2)$ vanishes, there exists a smooth $(2, 1)$ -form u on \mathbb{C}^2 such that $\bar{\partial}u = \tilde{f}$ on \mathbb{C}^2 . Then $v = u|_{\overline{\mathbb{H}^-}}$ is a smooth form on $\overline{\mathbb{H}^-}$ which satisfies $\bar{\partial}v = f$ on \mathbb{H}^- . □

Remark 3. Note that if we replace \mathbb{H}^- with the classical Hartogs triangle $\mathbb{T}^- = \mathbb{H}^- \cap \Delta \times \Delta$, where Δ is the unit disc in \mathbb{C} , then by Proposition 2.7 we have

$$H_{\mathbb{T}^-, \text{cur}}^{0,1}(\mathbb{C}^2) = H_{\mathbb{T}^-, L^2}^{0,1}(\mathbb{C}^2) = H_{\mathbb{T}^-, \infty}^{0,1}(\mathbb{C}^2) = 0.$$

Remark 4. We can also consider the classical Hartogs triangle \mathbb{T}^- as a domain in the bidisc $\Delta^2 = \Delta \times \Delta$, but now we have that both $H_{\mathbb{T}^-, \text{cur}}^{0,1}(\Delta^2)$ and $H_{\mathbb{T}^-, L^2}^{0,1}(\Delta^2)$ are infinite dimensional and $H_{\mathbb{T}^-, \infty}^{0,1}(\Delta^2) = 0$, since we can repeat the arguments used for the unbounded Hartogs triangle \mathbb{H}^- in \mathbb{C}^2 .

So for solving the $\bar{\partial}$ -equation with prescribed support in a noncompact complex manifold X such that $H_c^{0,1}(X) = 0$, which is the case for both \mathbb{C}^2 and Δ^2 , it is quite different to consider a relatively compact domain or a nonrelatively compact domain as support.

4. THE CASE OF THE HARTOGS TRIANGLES IN $\mathbb{C}\mathbb{P}^2$

In $\mathbb{C}\mathbb{P}^2$, we denote the homogeneous coordinates by $[z_0, z_1, z_2]$. On the domain where $z_0 \neq 0$, we set $z = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$. Let us define the domains \mathbb{H}^+ and \mathbb{H}^- by

$$\begin{aligned} \mathbb{H}^+ &= \{[z_0 : z_1 : z_2] \in \mathbb{C}\mathbb{P}^2 \mid |z_1| < |z_2|\}, \\ \mathbb{H}^- &= \{[z_0 : z_1 : z_2] \in \mathbb{C}\mathbb{P}^2 \mid |z_1| > |z_2|\}. \end{aligned}$$

Then $\mathbb{H}^+ \cap \mathbb{H}^- = \emptyset$ and $\overline{\mathbb{H}^+} \cup \overline{\mathbb{H}^-} = \mathbb{C}\mathbb{P}^2$. These domains are called Hartogs' triangles in $\mathbb{C}\mathbb{P}^2$. The Hartogs triangles provide examples of non-Lipschitz Levi-flat hypersurfaces (see [6]).

For $k \geq 0$ or $k = \infty$, we denote by $H_{\overline{\mathbb{H}^-}, \mathcal{C}^k}^{0,1}(\mathbb{C}\mathbb{P}^2)$ (resp., $H_{\overline{\mathbb{H}^-}, \text{cur}}^{0,1}(\mathbb{C}\mathbb{P}^2)$, $H_{\overline{\mathbb{H}^-}, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$) the Dolbeault cohomology group of bidegree $(0, 1)$ for \mathcal{C}^k -smooth forms (resp., currents, L^2 -forms) with support in $\overline{\mathbb{H}^-}$.

Again the vanishing of these groups means that one can solve the $\bar{\partial}$ equation with prescribed support in $\overline{\mathbb{H}^-}$ in the \mathcal{C}^k -smooth category (resp., the space of currents, the space of L^2 -forms).

We can also apply Propositions 2.4 and 2.10 for $\Omega = \mathbb{H}^-$ since $H^{0,1}(\mathbb{C}\mathbb{P}^2) = 0$, and we get the following.

Proposition 4.1. *We have, for $k \geq 0$ and for $k = \infty$, $H_{\overline{\mathbb{H}^-}, \mathcal{C}^k}^{0,1}(\mathbb{C}\mathbb{P}^2) = 0$ if and only if any holomorphic function on \mathbb{H}^+ which is \mathcal{C}^{k+1} -smooth on $\overline{\mathbb{H}^+}$ extends as a holomorphic function to $\mathbb{C}\mathbb{P}^2$.*

Proposition 4.2. *Any holomorphic function on \mathbb{H}^+ which is continuous on $\overline{\mathbb{H}^+}$ is constant.*

Proof. Let $f \in \mathcal{C}(\overline{\mathbb{H}^+}) \cap \mathcal{O}(\mathbb{H}^+)$. Notice that the boundary $b\mathbb{H}^+$ of \mathbb{H}^+ is foliated by a family of compact complex curves described in nonhomogeneous coordinates by

$$(4.1) \quad S_\theta = \{z = e^{i\theta} w\}, \quad \theta \in \mathbb{R}.$$

Restricted to each fixed θ , f is a continuous CR function on the compact Riemann surface S_θ . Thus f must be a constant on each S_θ . Since every Riemann surface S_θ contains the point $(0, 0)$, this implies that f must be constant on $b\mathbb{H}^+$. \square

Note that in the case of the unbounded Hartogs triangle in \mathbb{C}^2 , the function f needs to be of class \mathcal{C}^∞ on $\overline{\mathbb{H}^+}$ to be extendable as a holomorphic function to \mathbb{C}^2 (see Proposition 3.1 and the beginning of the proof of Theorem 3.5). But in $\mathbb{C}\mathbb{P}^2$, contrary to \mathbb{C}^2 we get the following (compare to Corollary 3.3 and Theorem 3.5) from the previous propositions.

Corollary 4.3. *For each $k \geq 0$, $H_{\overline{\mathbb{H}^-}, \mathcal{C}^k}^{0,1}(\mathbb{C}\mathbb{P}^2) = 0$, and $H_{\overline{\mathbb{H}^-}, \infty}^{0,1}(\mathbb{C}\mathbb{P}^2) = 0$.*

As in the case of \mathbb{C}^2 , we get the following for extendable currents.

Proposition 4.4. *Suppose that $H_{\overline{\mathbb{H}^-}, \text{cur}}^{0,1}(\mathbb{C}\mathbb{P}^2) = 0$. Then any holomorphic function on \mathbb{H}^+ which is extendable in the sense of currents is constant.*

Theorem 4.5. *$H_{\overline{\mathbb{H}^-}, \text{cur}}^{0,1}(\mathbb{C}\mathbb{P}^2)$ does not vanish and is Hausdorff.*

Proof. Let us consider the function h defined on the open subset \mathbb{H}^+ of $\mathbb{C}\mathbb{P}^2$ by $h([z_0 : z_1 : z_2]) = \frac{z_1}{z_2}$. It is holomorphic and bounded and hence defines an extendable current, but it is not constant, so by Proposition 4.4, we get $H_{\overline{\mathbb{H}^+}, \text{cur}}^{0,1}(\mathbb{C}\mathbb{P}^2) \neq 0$. By the Serre duality, to prove that $H_{\overline{\mathbb{H}^+}, \text{cur}}^{0,1}(\mathbb{C}\mathbb{P}^2)$ is Hausdorff, it is sufficient to prove that $H_{\infty}^{2,2}(\overline{\mathbb{H}^+}) = 0$.

Let f be a smooth $(2, 2)$ -form on $\overline{\mathbb{H}^+}$, and let U be a neighborhood of $\overline{\mathbb{H}^+}$. We can choose U such that \overline{U} is a connected proper subset of $\mathbb{C}\mathbb{P}^2$. Then f extends as a smooth $(2, 2)$ -form on U , called \tilde{f} . By Malgrange’s theorem, the top degree Dolbeault cohomology group $H^{2,2}(U)$ vanishes since U is a noncompact connected complex manifold. Thus there exists a smooth $(2, 1)$ -form u on U such that $\bar{\partial}u = \tilde{f}$ on U . Then $v = u|_{\overline{\mathbb{H}^+}}$ is a smooth form on $\overline{\mathbb{H}^+}$ which satisfies $\bar{\partial}v = f$ on \mathbb{H}^+ . \square

Let us now consider the L^2 Dolbeault cohomology with prescribed support in a Hartogs triangle in $\mathbb{C}\mathbb{P}^2$. As usual we endow \mathbb{H}^+ with the restriction of the Fubini–Study metric of $\mathbb{C}\mathbb{P}^2$. The following proposition was already proved in [4, Proposition 6].

Proposition 4.6. *Let $\mathbb{H}^+ \subset \mathbb{C}\mathbb{P}^2$ be the Hartogs triangle. Then we have the following:*

- (1) *The Bergman space of L^2 holomorphic functions $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ on the domain \mathbb{H}^+ separates points in \mathbb{H}^+ .*
- (2) *There exist nonconstant functions in the space $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$. However, this space does not separate points in \mathbb{H}^+ and is not dense in the Bergman space $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$.*
- (3) *Let $f \in W^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ be a holomorphic function on \mathbb{H}^+ which is in the Sobolev space $W^2(\mathbb{H}^+)$. Then f is a constant.*

Proposition 4.7. *Let $\mathbb{H}^+ \subset \mathbb{C}\mathbb{P}^2$ be the Hartogs triangle. Any function $f \in W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ can be extended to a function in $W^1(\mathbb{C}\mathbb{P}^2)$.*

Proof. In the nonhomogeneous holomorphic coordinates (z, w) for \mathbb{H}^+ , any function $f \in W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ has the form (see [4, Proposition 6])

$$f_k(z, w) = \left(\frac{z}{w}\right)^k, \quad k \in \mathbb{N}.$$

It suffices to prove the proposition for each $f_k(z, w)$.

Let $\chi(t) \in C^\infty(\mathbb{R})$ be a function defined by $\chi(t) = 0$ if $t \leq 0$, and $\chi(t) = 1$ if $t \geq 1$. Let \tilde{f}_k be the function defined by

$$(4.2) \quad \tilde{f}_k(z, w) = \chi\left(1 + \frac{1}{3}\left(1 - \frac{|z|^2}{|w|^2}\right)\right) f_k(z, w).$$

On $|z| < |w|$, it is easy to see that $\tilde{f}_k = f_k$. Thus \tilde{f}_k is an extension of f_k to $\mathbb{C}\mathbb{P}^2$.

To see that \tilde{f}_k is in $W^1(\mathbb{C}\mathbb{P}^2)$, we first note that the function

$$\chi\left(1 + \frac{1}{3}\left(1 - \frac{|z|^2}{|w|^2}\right)\right) = 0$$

when restricted to $\{|z| \geq 2|w|\}$. Thus it is supported in $\{|z| \leq 2|w|\}$. On its support, the function $\frac{|z|}{|w|}$ is bounded. Using this fact and differentiating under the

chain rule, we have

$$(4.3) \quad |\nabla\chi \left(1 + \frac{1}{3}\left(1 - \frac{|z|^2}{|w|^2}\right)\right)| \leq C(\sup|\chi'|)\frac{1}{|w|} \leq C\frac{1}{|w|}.$$

Repeating the arguments as before, we see that the function $\frac{1}{|w|}$ is in L^2 on $\{|z| \leq 2|w|\}$. Since the function f_k is bounded on the set $\{|z| \leq 2|w|\}$, we conclude from 4.3 the partial derivatives of \tilde{f}_k are in $L^2(\mathbb{C}\mathbb{P}^2)$. Thus \tilde{f}_k is an extension in $W^1(\mathbb{C}\mathbb{P}^2)$ of f_k . \square

Remark 5. Suppose that D is a bounded domain with Lipschitz boundary. Then any function $f \in W^1(D)$ extends as a function in $W^1(\mathbb{C}\mathbb{P}^2)$. It is not known if this is true for the Hartogs triangle \mathbb{H}^+ . In the proof of Proposition 4.7, we have used the fact that the function f_k is in $W^1(\mathbb{H}^+)$ and is bounded on \mathbb{H}^+ .

Theorem 4.8. *Let $\mathbb{H}^- \subset \mathbb{C}\mathbb{P}^2$ be the Hartogs triangle. Then the cohomology group $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2) \neq 0$ and is infinite dimensional.*

Proof. We recall that $\mathbb{H}^+ = \mathbb{C}\mathbb{P}^2 \setminus \overline{\mathbb{H}^-}$. From Proposition 4.6, the space of holomorphic functions in $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ is infinite dimensional. In the nonhomogeneous coordinates, consider the holomorphic functions of the type $f_k = (\frac{z}{w})^k, k \in \mathbb{N}$.

We define the operator $\bar{\partial}_{\tilde{c}}$ as the weak minimal realization of $\bar{\partial}$. Then the domain of $\bar{\partial}_{\tilde{c}}$ is the space of L^2 -forms f in $\mathbb{C}\mathbb{P}^2$ with support in $\overline{\mathbb{H}^-}$ such that $\bar{\partial}f$ is also an L^2 -form in $\mathbb{C}\mathbb{P}^2$.

Using Proposition 4.7, each holomorphic function f_k can be extended to a function $\tilde{f}_k \in W^1(\mathbb{C}\mathbb{P}^2)$. Suppose that $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2) = 0$. Then we can solve $\bar{\partial}_{\tilde{c}}u_k = \bar{\partial}\tilde{f}_k$ in $\mathbb{C}\mathbb{P}^2$ with prescribed support for u_k in $\overline{\mathbb{H}^-}$. Let $H_k = \tilde{f}_k - u_k$. Then H_k is a holomorphic function in $\mathbb{C}\mathbb{P}^2$, and hence a constant. But $H_k = f_k$ on \mathbb{H}^+ , a contradiction. This implies that the space $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$ is nontrivial.

Next we prove that $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$ is infinite dimensional. Each function \tilde{f}_k corresponds to a (0,1)-form $\bar{\partial}\tilde{f}_k$. We set $g_k = \bar{\partial}\tilde{f}_k$. Then g_k is in $\text{Dom}(\bar{\partial}_{\tilde{c}})$ and satisfies $\bar{\partial}_{\tilde{c}}g_k = 0$. Thus it induces an element $[g_k]$ in $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$. To see that $[g_k]$'s are linearly independent, let $N > 1$ be a positive integer and $F_N = \sum_{k=1}^N c_k f_k$, where c_k are constants. Set $G_N = \sum_{k=1}^N c_k g_k$. Suppose that $[G_N] = 0$. Then we can solve $\bar{\partial}_{\tilde{c}}u = G_N$, and the function F_N holomorphic in \mathbb{H}^+ extends holomorphically to $\mathbb{C}\mathbb{P}^2$. Thus F_N must be a constant and $c_1 = \dots = c_N = 0$. Thus $[g_k]$'s are linearly independent. This proves that $H_{\mathbb{H}^-, L^2}^{0,1}(\mathbb{C}\mathbb{P}^2)$ is infinite dimensional. \square

Remark 6. It follows from Proposition 2.1 and Theorem 4.8 that $H_{\mathbb{H}^-, \text{cur}}^{0,1}(\mathbb{C}\mathbb{P}^2)$ is also infinite dimensional.

Lemma 4.9. *The range of the strong L^2 closure of $\bar{\partial}$,*

$$(4.4) \quad \bar{\partial}_s : L_{2,1}^2(\mathbb{H}^-) \rightarrow L_{2,2}^2(\mathbb{H}^-),$$

is closed and equal to $L_{2,2}^2(\mathbb{H}^-)$.

Proof. It is clear that $\bar{\partial}$ has closed range in the top degree, and the range is $L_{2,2}^2(\mathbb{H}^-)$. Let $f \in L_{2,2}^2(\mathbb{H}^-)$. We extend f to be 0 outside \mathbb{H}^- . Let U be an open neighborhood of $\overline{\mathbb{H}^-}$. Then f is in $L_{2,2}^2(U)$. We can choose U such that \bar{U} is

a proper subset of $\mathbb{C}\mathbb{P}^2$ and U has Lipschitz boundary. Since one can solve the $\bar{\partial}$ equation for top degree forms on U , there exists $u \in L^2_{2,1}(U)$ such that

$$\bar{\partial}u = f$$

in the weak sense.

It suffices to show that f is in the range of $\bar{\partial}_s$. Since U has Lipschitz boundary, using Friedrichs's lemma, there exists a sequence $u_\nu \in C^\infty(\bar{U})$ such that $u_\nu \rightarrow u$ and $\bar{\partial}u_\nu \rightarrow f$ in $L^2_{2,2}(U)$. Restricting u_ν to \mathbb{H}^- , we find that u is in the domain of $\bar{\partial}_s$ and that

$$\bar{\partial}_s u = f.$$

Thus the range of $\bar{\partial}_s$ is equal to $L^2_{2,2}(\mathbb{H}^-)$. The lemma is proved. □

Corollary 4.10. *The cohomology group $H^{0,1}_{\mathbb{H}^-,L^2}(\mathbb{C}\mathbb{P}^2)$ is Hausdorff and infinite dimensional.*

Theorem 4.11. *Let us consider the Hartogs triangle $\mathbb{H}^- \subset \mathbb{C}\mathbb{P}^2$. Then the cohomology group $H^{2,1}_{\bar{\partial}_s,L^2}(\mathbb{H}^-)$ is infinite dimensional.*

Proof. Suppose that $\bar{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \rightarrow L^2_{2,1}(\mathbb{H}^-)$ does not have closed range. Then $H^{2,1}_{\bar{\partial}_s,L^2}(\mathbb{H}^-)$ is non-Hausdorff, and hence infinite dimensional.

Suppose that $\bar{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \rightarrow L^2_{2,1}(\mathbb{H}^-)$ has closed range. Using Lemma 4.9, $\bar{\partial}_s : L^2_{2,1}(\mathbb{H}^-) \rightarrow L^2_{2,2}(\mathbb{H}^-)$ has closed range. From the L^2 Serre duality, $\bar{\partial}_{\bar{c}} : L^2(\mathbb{H}^-) \rightarrow L^2_{0,1}(\mathbb{H}^-)$ and $\bar{\partial}_{\bar{c}} : L^2_{0,1}(\mathbb{H}^-) \rightarrow L^2_{0,2}(\mathbb{H}^-)$ both have closed range. Furthermore,

$$(4.5) \quad H^{2,1}_{\bar{\partial}_s,L^2}(\mathbb{H}^-) \cong H^{0,1}_{\mathbb{H}^-,L^2}(\mathbb{C}\mathbb{P}^2).$$

Thus from Theorem 4.8, it is infinite dimensional. □

Remark 7.

- (1) Let $\mathbb{T} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2| < |z_1| < 1\}$ be the Hartogs triangle in \mathbb{C}^2 . Then by Proposition 2.7,

$$H^{0,1}_{\bar{\partial}_c,L^2}(\mathbb{T}) = H^{0,1}_{\mathbb{T},L^2}(\mathbb{C}^2) = 0.$$

This is in sharp contrast to Corollary 4.10.

It is well known that $H^{0,1}(\mathbb{T}) = 0$ since \mathbb{T} is pseudoconvex, but $H^{0,1}(\bar{\mathbb{T}})$ (cohomology with forms smooth up to the boundary) is infinite dimensional (see [16]). In fact, $H^{0,1}(\bar{\mathbb{T}})$ is even non-Hausdorff (see [12]). We also refer the reader to the recent survey paper on the Hartogs triangle [15].

- (2) If D is a domain in $\mathbb{C}\mathbb{P}^n$ with C^2 boundary, then we have L^2 existence theorems for $\bar{\partial}$ on D for all degrees (see [1], [6], [2]). This follows from the existence of bounded plurisubharmonic functions on pseudoconvex domains in $\mathbb{C}\mathbb{P}^n$ with C^2 boundary (see [13]). This is even true if D has only Lipschitz boundary (see [5]).
- (3) Suppose that D is a pseudoconvex domain in $\mathbb{C}\mathbb{P}^n$ with Lipschitz boundary. We have $H^{p,q}_{L^2}(D) = 0$ for all $q > 0$. By the L^2 Serre duality (see [4]), we have $H^{0,1}_{\bar{\partial}_c,L^2}(D) = H^{0,1}_{D,L^2}(\mathbb{C}\mathbb{P}^n) = 0$. Corollary 4.10 shows that the Lipschitz condition cannot be removed.

- (4) From a result of Takeuchi [17], \mathbb{H}^- is Stein. It is well known that for any p , $0 \leq p \leq 2$, $\bar{\partial} : L^2_{p,0}(\mathbb{H}^-, \text{loc}) \rightarrow L^2_{p,1}(\mathbb{H}^-, \text{loc})$ has closed range (see [8]), and the cohomology $H^p_{L^2_{\text{loc}}}(\mathbb{H}^-)$ in the Fréchet space $L^2_{0,1}(\mathbb{H}^-, \text{loc})$ is trivial.
- (5) The (weak) L^2 theory holds for any pseudoconvex domain without any regularity assumption on the boundary for $(0,1)$ -forms. The (weak) L^2 Cauchy–Riemann operator $\bar{\partial} : L^2(\mathbb{H}^-) \rightarrow L^2_{0,1}(\mathbb{H}^-)$ has closed range and $H^{0,1}_{L^2}(\mathbb{H}^-) = 0$ (see [6] or [2]).
- (6) For $p = 1$ or $p = 2$, it is not known if the Cauchy–Riemann operator $\bar{\partial} : L^2_{p,0}(\mathbb{H}^-) \rightarrow L^2_{p,1}(\mathbb{H}^-)$ has closed range. It is also not known if $\bar{\partial}$ in the weak sense is equal to $\bar{\partial}_s$.
- (7) It is not known if the strong L^2 Cauchy–Riemann operator $\bar{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \rightarrow L^2_{2,1}(\mathbb{H}^-)$ has closed range.

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