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Bounded Plurisubharmonic Exhaustion Functions and Levi-flat Hypersurfaces

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In Memory of Professor Qikeng Lu (1927–2015)

Abstract In this paper, we survey some recent results on the existence of bounded plurisubharmonic functions on pseudoconvex domains, the Diederich–Fornæss exponent and its relations with existence of domains with Levi-flat boundary in complex manifolds.

Keywords Plurisubharmonic exhaustion functions, Levi-flat hypersurface, the Cauchy–Riemann operator

MR(2010) Subject Classification 32T35, 32V40

1 Introduction

Any Stein domain Ω can be exhausted by a smooth strictly plurisubharmonic function. It is natural to ask if there exists a *bounded* strictly plurisubharmonic exhaustion function for Ω. Existence of Hölder continuous bounded plurisubharmonic exhaustion functions for pseudoconvex domains with C^2 boundary in $\mathbb{C}^n$ (or more generally in a Stein manifold) is proved by Diederich and Fornaess: For any bounded pseudoconvex domain Ω with C^2 boundary in a Stein manifold, there exist a positive constant η and a defining function r of Ω such that $\hat{r} = -(-r)^\eta$ is plurisubharmonic on Ω ([16]; see also [33]). This result of Diederich and Fornæss was generalized to bounded pseudoconvex domains with C^1 boundary by Kerzman and Rosay [26] and those with Lipschitz boundary by Demailly [15] and Harrington [22]. The constant η is called a Diederich–Fornæss exponent. The supremum of all Diederich–Fornæss exponents is called the Diederich–Fornæss index of Ω. A similar result for pseudoconvex domains in $\mathbb{CP}^n$ is proved by Ohsawa and Sibony [31]. This result has been extended to pseudoconvex domains in $\mathbb{CP}^n$ with Lipschitz boundary recently by Harrington in [23]. Existence of bounded plurisubharmonic exhaustion functions has many applications in holomorphic function theory on pseudoconvex

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domains. The Diederich–Fornæss index has implications in regularity theory in the $\bar{\partial}$-Neumann problem.

In this paper, we review and elaborate on some recent results on the existence of bounded plurisubharmonic functions on pseudoconvex domains, the Diederich–Fornæss exponent and its relations with existence of domains with Levi-flat boundary in complex manifolds. Our plan of the paper is as follows. We first discuss the relations between strong Oka's lemma and bounded plurisubharmonic exhaustion functions. In Section 3, we examine the Diederich–Fornæss exponent and L^2-theory of the $\bar{\partial}$-Neumann Laplacian for pseudoconvex domains with Lipschitz boundary in complex Kähler manifolds with nonnegative curvature. In Section 4, an example of a pseudoconvex Stein domain with smooth boundary is given such that there exist no bounded plurisubharmonic functions. The domain has real-analytic Levi-flat boundary and the $\bar{\partial}$-operator does not have closed range for some degree. This is in sharp contrast to the case of bounded pseudoconvex domains in Stein manifolds. Finally, we show that the Diederich–Fornæss exponent is closely related to the non-existence of Levi-flat hypersurfaces; yet bounded plurisubharmonic exhaustions functions do exist on some domains with Levi-flat boundary.

2 The Strong Oka's Lemma and Bounded Plurisubharmonic Exhaustion Functions

The classical Oka's lemma states that if Ω is a pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, then $-\log\delta$ is plurisubharmonic where δ is some distance function to the boundary. Let M be a complex hermitian manifold with the metric form ω. Let Ω be relatively compact pseudoconvex domain in M. We say that a distance function δ to the boundary $b\Omega$ satisfies the strong Oka condition if it can be extended from a neighborhood of $b\Omega$ to Ω such that δ satisfies

$$\mathrm{i}\partial\bar{\partial}(-\log\delta) \geq c_0\omega \quad \text{in } \Omega \tag{2.1}$$

for some constant $c_0 > 0$.

For a bounded pseudoconvex domain Ω with C^2 boundary in $\mathbb{C}^n$ or in a Stein manifold, Diederich–Fornæss [16] shows that there exists a Hölder continuous strictly plurisubharmonic exhaustion function with Hölder exponent $0 < \eta < 1$. The existence of such bounded plurisubharmonic functions is closely related to the existence of a distance function satisfying the strong Oka's condition.

Lemma 2.1 *Let M be a complex hermitian manifold with metric ω and let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2 boundary. Suppose that the distance function δ satisfies the strong Oka condition* (2.1). *Then the following two conditions are equivalent*:

(1) *For* $0 < t_0 \leq 1$,

$$\mathrm{i}\partial\bar{\partial}(-\delta^{t_0}) \geq 0.$$

(2) *For any* $0 < t < t_0$, *there exists some constant* $C_t > 0$ *such that*

$$\mathrm{i}\partial\bar{\partial}(-\delta^t) \geq C_t\delta^t\left(\omega + \mathrm{i}\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}\right).$$

The lemma follows from Lemma 2.2 in Cao–Shaw–Wang [9].

Lemma 2.2 *Let M be a complex hermitian manifold and let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2 boundary $b\Omega$. Let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$ with respect to the hermitian metric ω. The following conditions are equivalent*:

(1) *There exists $0 < t_0 \le 1$ such that for any $0 < t < t_0$, there exists some constant $C_t > 0$ satisfying*

$$\mathrm{i}\partial\bar{\partial}(-\delta^t) \ge C_t \delta^t \Big(\omega + \mathrm{i}\frac{\partial\delta \wedge \bar{\partial}\delta}{\delta^2}\Big).$$

(2) *The distance function δ satisfies the strong Oka's condition*

$$\mathrm{i}\partial\bar{\partial} - \log\delta \ge C\omega$$

for some constant $C > 0$.

Proof That (2) implies (1) is proved in Proposition 2.3 in Cao–Shaw [11]. To show that (1) implies (2), we use

$$\mathrm{i}\partial\bar{\partial}(-\log\delta^t) = \mathrm{i}\frac{\partial\bar{\partial}(-\delta^t)}{\delta^t} + \frac{\mathrm{i}\partial\delta^t \wedge \bar{\partial}\delta^t}{\delta^{2t}}.$$

From assumption (1), we have

$$\mathrm{i}\partial\bar{\partial}(-\log\delta^t) \ge C_t\omega.$$

This proves (2) with $C = C_t$. □

Theorem 2.3 (Diederich–Fornæss) *Let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2 boundary in a Stein manifold M. Then there exists a defining function ρ and some number $0 < t < 1$ such that $\tilde{\delta} = -(-\rho)^t$ is a strictly plurisubharmonic bounded exhaustion function on Ω.*

If Ω is a Lipschitz bounded pseudoconvex domain in a Stein manifold, it is proved in Demailly [15] that there exists a bounded strictly plurisubharmonic function in Ω (see also Kerzman–Rosay [26] for the C^1 case). It is also true for pseudoconvex domains with Lipschitz boundary in $\mathbb{C}P^n$ (see Harrington [23]). We also remark that strictly plurisubharmonic bounded exhaustion functions might not exist if the Lipschitz boundary (as a graph) condition is dropped (see [16]).

When the complex manifold is Kähler with positive curvature, the following result is proved by Ohsawa–Sibony [31] for domains with C^2 smoothness.

Theorem 2.4 (Ohsawa–Sibony) *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2 boundary $b\Omega$ and let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$ with the Fubini–Study metric ω. Then there exists $0 < t < 1$ such that $-\delta^t$ is a strictly plurisubharmonic bounded exhaustion function on Ω.*

For a pseudoconvex domain in $\mathbb{C}P^n$ with only Lipschitz boundary, it is shown that the distance function with respect to the Fubini–Study metric may not be strictly plurisubharmonic (see [23]). But one can construct a different distance function which can be used to yield a Hölder continuous strictly plurisubharmonic exhaustion function. This is proved more recently in [23] for domains with Lipschitz boundary. The existence of bounded plurisubharmonic functions have implication in the existence and regularity for the $\bar{\partial}$ equation, which we will discuss in the next section.

3 The Diederich–Fornæss Exponent and the $\bar\partial$-Equation

Let X be a Kähler manifold with Kähler metric ω. Let Ω be a relatively compact domain in X with C^2-smooth boundary $b\Omega$. Let $\rho(z)$ be the signed distance function from z to $b\Omega$ such that $\rho(z) = -d(z, b\Omega)$ for $z \in \Omega$ and $\rho(z) = d(z, b\Omega)$ when $z \in X \setminus \Omega$. Let φ be a real-valued C^2 function on Ω. Let $L^2_{p,q}(\Omega, \mathrm{e}^{-\varphi})$ be the space of (p, q)-forms u on Ω such that

$$\|u\|^2_\varphi = \int_\Omega |u|^2_\omega \mathrm{e}^{-\varphi} dV < \infty.$$

We will also use $(\cdot, \cdot)_\varphi$ to denote the associated inner product. Let $\bar\partial^*_\varphi$ be the adjoint of the maximally defined $\bar\partial\colon L^2_{p,q}(\Omega, \mathrm{e}^{-\varphi}) \to L^2_{p,q+1}(\Omega, \mathrm{e}^{-\varphi})$. We now recall an integration by parts formula due to Morrey, Kohn, and Hörmander that is basic to the study of the complex Laplacian. With the above notations, we can now state the Bochner–Kodaira–Morrey–Kohn–Hörmander formula: For any $u \in C^1_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar\partial^*)$, we have

$$\|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi = \|\overline{\nabla} u\|^2_\varphi + +(\Theta u,\ u)_\varphi + ((\partial\bar\partial\varphi)u,\ u)_\varphi + \int_{b\Omega} \langle(\partial\bar\partial\rho)u,\ u\rangle \mathrm{e}^{-\varphi} dS, \qquad (3.1)$$

where dS is the induced surface element on $b\Omega$. Herein, for convenience, we use the notations

$$|\overline{\nabla} u|^2 = \sum_{j=1}^n |\nabla_{\bar L_j} u|^2 \quad \text{and} \quad |\overline{\overline{\nabla}} u|^2 = \sum_{j=1}^n |\nabla_{L_j} u|^2.$$

Note that in the integrand of the last term in (3.1), the local frame field $L_1, \ldots, L_n$ are chosen so that $L_n = \sqrt{2}(\partial\rho)^*$, where $(\partial\rho)^*$ is the dual vector of the $(1, 0)$-form $\partial\rho$.

The following proposition is a simple application of a result of Berndtsson and Charpentier [6, Theorem 2.3] (compare [9, 24]).

Proposition 3.1 *Let (X, ω) be a Kähler manifold of dimension n. Assume that the curvature operator Θ is semi-positive on (p, q)-forms for all $1 \le q \le n$. Let Ω be a Stein domain in X. Suppose that there exist a distance function $\rho < 0$ and a constant $\eta > 0$ such that*

$$-\mathrm{i}\partial\bar\partial(-\rho)^\eta \ge \eta K(-\rho)^\eta \omega$$

on Ω for some constant $K > 0$. Then the $\bar\partial$-Neumann Laplacian $\Box$ has a bounded inverse N on $L^2_{p,q}(\Omega)$ and for $u \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$,

$$\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \ge \frac{q\eta K}{4} \|u\|^2. \qquad (3.2)$$

Furthermore, the operator N is bounded from $W^{\frac{\eta}{2}}_{p,q}(\Omega) \to W^{\frac{\eta}{2}}_{p,q}(\Omega)$ with

$$\|\bar\partial^* N u\|^2_{\frac{\eta}{2}} \le C_\eta \|u\|^2_{\frac{\eta}{2}}; \quad \|\bar\partial N u\|^2_{\frac{\eta}{2}} \le C_\eta \|u\|^2_{\frac{\eta}{2}} \qquad (3.3)$$

for any $u \in W^{\frac{\eta}{2}}_{p,q}(\Omega)$.

Let ω_{FS} be the Kähler form associated with the Fubini–Study metric on $\mathbb{CP}^n$. Let Ω be a (proper) pseudoconvex domain in $\mathbb{CP}^n$ with C^2-smooth boundary. Let $\delta(z) = d(z, b\Omega)$ be the distance, with respect to the Fubini–Study metric, from z to the boundary $b\Omega$. Let $\Omega_\varepsilon = \{z \in \Omega \mid \delta(z) > \varepsilon\}$. It then follows from Takeuchi's theorem [35] that there exists a universal constant $K_0 > 0$ such that

$$\mathrm{i}\partial\bar\partial(-\log\delta) \ge K_0 \omega_{\mathrm{FS}} \qquad (3.4)$$

on Ω. In particular, there exists $\epsilon_0 > 0$ such that

$$\partial\bar{\partial}(-\delta)(\zeta,\overline{\zeta}) \geq K_0\epsilon|\zeta|^2_{\omega_{\rm FS}} \tag{3.5}$$

for all $\zeta \in T^{1,0}_x(b\Omega_\epsilon)$ for $0 \leq \epsilon \leq \epsilon_0$. (See [10, 21] for different proofs of Takeuchi's theorem.) Obsawa and Sibony [31] showed — as a consequence of Takeuchi's theorem — that, there exists $0 < \eta \leq 1$ such that

$$\mathrm{i}\partial\bar{\partial}(-\delta^\eta) \geq K\eta\delta^\eta\omega_{\rm FS} \tag{3.6}$$

on Ω for some constant $K > 0$. (See [10, Proposition 2.3] and [9, Lemma 2.2] for a more streamlined proof of this fact.) Such a constant η_0 is called a Diederich–Fornæss exponent of Ω. We refer the reader to [23] for similar results when the boundary is only Lipschitz. We then have:

Proposition 3.2 *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then for $0 \leq p \leq n$ and $1 \leq q < n$, the $\bar{\partial}$-Neumann Laplacian $\Box$ has a bounded inverse N on $L^2_{p,q}(\Omega)$. Furthermore, we have N, $\bar{\partial}^*N$ $\bar{\partial}N$ and the Bergman projection $B = I - \bar{\partial}^*N\bar{\partial}$ are all exact regular on $W^s_{p,q}(\Omega)$ for all $s < \frac{\eta_0}{2}$.*

4 Nonexistence of Bounded Plurisubharmonic Functions on Some Stein Domains

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that Ω has Lipschitz boundary. Then there exists a bounded Hölder continuous plurisubharmonic exhaustion function. However, if we drop the assumption that Ω is Lipschitz, then this is no longer true. Let

$$\mathbb{H} = \{(z,w) \mid |z| < |w| < 1\}$$

be the Hartogs triangle in $\mathbb{C}^2$. Then $\mathbb{H}$ admits no bounded plurisubharmonic exhaustion function. Notice that the Hartogs triangle is not Lipschitz in the sense that it is not the graph of a Lipschtiz function near $(0,0)$.

In a complex manifold $\mathcal{X}$, the situation is different. We will give an example of a Stein domain with smooth boundary in a complex manifold which does not have a plurisubharmonic exhaustion function.

Let $\alpha > 1$ be a real number and let Γ be the subgroup of $\mathbb{C}^*$ generated by $\alpha > 0$. We will take $\alpha = \mathrm{e}^{2\pi}$. Let $T = \mathbb{C}^*/\Gamma$ be the torus. Let

$$\mathcal{X} = \mathbb{C}P^1 \times T$$

be equipped with the product metric ω from the Fubini–Study metric for $\mathbb{C}P^1$ and the flat metric for T. Let D_∞ be the domain in $\mathcal{X}$ defined by

$$D_\infty = \{(z,[w]) \in \mathbb{C}P^1 \times T \mid \mathrm{Re}\, zw > 0\}, \tag{4.1}$$

where z is the inhomogeneous coordinate on $\mathbb{C}P^1$. The domain D_∞ is biholomorphic to the product domain Ω in $\mathbb{C}^2$ where

$$\Omega = \mathbb{C}^* \times A = \mathbb{C}^* \times \{w \in \mathbb{C} \mid \mathrm{e}^{-\frac{\pi}{2}} < |w| < \mathrm{e}^{\frac{\pi}{2}}\}$$

via the map $\Phi : \Omega \to D_\infty$ defined by

$$\Phi : (z,w) \to (z, [z^{-1}\mathrm{e}^{\mathrm{i}\log w}]).$$

Such domain was first introduced by [29]. It is also related to a family of domains studied by [5]. Notice that the domain D_∞ has Levi-flat boundary. Furthermore, it has the following properties

- The boundary of D_∞ is smooth and real-analytic and Levi-flat.
- D_∞ is Stein.
- The boundary bD_∞ has two tori $T_0 = \{0\} \times T$ and $T_\infty = \{\infty\} \times T$ which decompose the boundary into two disjoint parts.

Proposition 4.1 *There exists no non-constant bounded pluri-subharmonic exhaustion function on $\overline{D_\infty}$.*

Proof Suppose that there exists a bounded continuous plurisubharmonic function $\phi : \overline{D}_\infty \to (-L, 0]$, where $L > 0$. Then we parametrize D_∞ by $(z, w) \in \mathbb{C}^* \times A$ as before. For each fixed w, we have that ϕ is a continuous bounded subharmonic function in $\mathbb{C}^*$, this implies that ϕ is constant in z.

On the other hand, $\phi = 0$ on the boundary and each (z, w) will pass through the boundary points in T_0 or T_∞, this implies that $\phi = 0$ on $\overline{D}_\infty$. For details of the proof, see [13]. □

Remark 4.2 The defining function Re zw for D_∞ is pluriharmonic (hence plurisubharmonic) but it is not a global defining function on D_∞ since it is not defined near T_∞. We also remark that an earlier related result has been proved in [31].

Remark 4.3 Recall that a domain is called hyperconvex if there exists a bounded continuous plurisubharmonic exhaustion function. The domain D_∞ is not hyperconvex.

Proposition 4.4 *The range of $\bar\partial : L^2_{2,0}(D_\infty) \to L^2_{2,1}(D_\infty)$ is not closed. In particular, the space $H^{2,1}_{L^2}(D_\infty)$ is not Hausdorff.*

This is proved in [13] using L^2 Serre duality [12].

5 The Diederich–Fornæss Exponent and Domains with Levi-flat Boundary

The Diederich–Fornæss index is also related to non-existence of Stein domains with Levi-flat boundaries in complex manifolds (see [20] and [3]).

Theorem 5.1 *Let Ω be a bounded Stein domain with C^2 boundary in a complex manifold M of dimension n. If the Diederich–Fornæss index of Ω is greater than k/n, $1 \le k \le n-1$, then Ω has a boundary point at which the Levi form has rank $\ge k$.*

In particular, we have the following corollary.

Corollary 5.2 *If the Diederich–Fornæss index is greater than $1/n$, then its boundary cannot be Levi flat; and if the Diederich–Fornæss index is greater than $1 - 1/n$, then its boundary must have at least one strongly pseudoconvex boundary point.*

Our results above are inspired by the work of Nemirovskii who showed that any smooth bounded Stein domain with a defining function that is plurisubharmonic on the domain cannot have Levi-flat boundary ([28, Corollary]).

When $n = 2$, this is best possible in the sense that there exists a Stein domain with Levi-flat boundary with Diederich–Fornæss exponent equal to $\frac{1}{2}$.

The domain Ω is given by the example of [18]. It can be described as follows: Let Σ be a compact Riemann surface of genus $g \geq 2$. Then we can write

$$\Sigma = D/\Gamma$$

where D is the unit disc and Γ is a discrete subgroup of $\mathrm{Aut}(D)$. A linear fractional transformation

$$T = \frac{az+b}{cz+d}$$

acts on $\mathbb{C}P^1$ via

$$T\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

in homogeneous coordinates. Thus Γ can also be considered as a subgroup of $\mathrm{Aut}(\mathbb{C}P^1)$.

Let $U_1 = \{[z_1, z_2] \in \mathbb{C}P^1 \mid z_2 \neq 0\}$ and $\phi_1 : U_1 \to \mathbb{C}$ given by

$$\phi([z_1, z_2]) = \frac{z_1}{z_2}.$$

Denote $\widehat{D} = \phi_1^{-1}(D) \subset \mathbb{C}P^1$. Let

$$X = D \times \mathbb{C}P^1/\sim_\Gamma,$$

where $(z, \zeta) \sim_\Gamma (z', \zeta')$ iff there exists some $\gamma \in \Gamma$ such that $\gamma z = z'$ and $\gamma\zeta = \zeta'$. Let

$$\Omega = D \times \widehat{D}/\sim_\Gamma$$

and let $\widetilde{\Omega}$ be the complement of Ω in X. Note that both Ω and $\widetilde{\Omega}$ share the Levi-flat boundary

$$\partial\Omega = \partial\widetilde{\Omega} = D \times \partial\widehat{D}/\sim_\Gamma .$$

Proposition 5.3 *The domain Ω has Diederich–Fornæss index $1/2$ in the weak sense*[1)]*. However, Ω is not Stein.*

Proof The theorem is implicit in [18] (see also [1]). We provide a proof for the reader's convenience. Let (z, ζ) be the coordinates for $D \times U_1$. Define

$$\rho(z, \zeta) = \left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|^2 - 1. \tag{5.1}$$

Since $\rho(z, \zeta)$ is invariant under $\mathrm{Aut}(D)$. It can be pushed down to X and regarded as defining function for Ω. It follows from straightforward computations that

$$\begin{aligned}\partial\bar{\partial}(-\log(-\rho)) - \eta\frac{\partial\rho \wedge \bar{\partial}\rho}{\rho^2} &= \left(1 - \eta\left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|^2\right)\left(\frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} + \frac{d\zeta \wedge d\bar{\zeta}}{(1-|\zeta|^2)^2}\right) \\ &\quad + \left(1 - \eta\frac{|z-\zeta|^2}{(1-|z|^2)(1-|\zeta|^2)}\right)\left(\frac{dz \wedge d\bar{\zeta}}{(1-z\bar{\zeta})^2} + \frac{d\zeta \wedge d\bar{z}}{(1-\bar{z}\zeta)^2}\right).\end{aligned}$$

The above expression (consider as a Hermitian form) is non-negative if and only if its determinant is non-negative. A direct computation yields that the determinant equals to

$$\frac{|z-\zeta|^2(2(1-|z|^2)(1-|\zeta|^2) + (1-2\eta)|z-\zeta|^2)}{(1-|z|^2)^2(1-|\zeta|^2)^2|1-\bar{z}\zeta|^4},$$

1) A constant $0 < \eta < 1$ is a Diederich–Fornæss exponent for a domain Ω in weak sense if one only requires $-(-\rho)^\eta$ to be plurisubharmonic on Ω where ρ is a defining function of Ω; it is a Diederich–Fornæss exponent in strong sense if one requires $-(-\rho)^\eta$ to be strongly plurisubharmonic.

which is non-negative for all $(z,\zeta) \in D \times D$ if and only if $\eta \leq 1/2$. This implies that the Diederich–Fornæss exponent η for Ω is greater than or equal to 1/2. From Corollary 5.2, we conclude that the Diederich–Fornæss index in the weak sense for Ω is 1/2. Note that $\Omega \supset \{(z,z) \in D \times D\}/\sim_\Gamma$, a compact Riemann surface. Thus Ω is not Stein. □

We refer the reader to interesting recent results by Adachi [2] on the weighted Bergman kernel of Ω. As far as we know, whether or not the range of $\bar\partial\colon L^2(\Omega) \to L^2_{0,1}(\Omega)$ is closed remains open.

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