Groups That Split Over Subgroups

Annie Holden

October 27, 2022

Amalgams

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Definition

If we have $f : C \hookrightarrow A$ and $g : C \hookrightarrow B$ then

$$A *_C B = \langle S_A, S_B \mid R_A, R_B, f(c)g^{-1}(c) = 1 \rangle$$

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Example

- $SL_2(\mathbb{Z}) \cong \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Amalgams As Fundamental Groups

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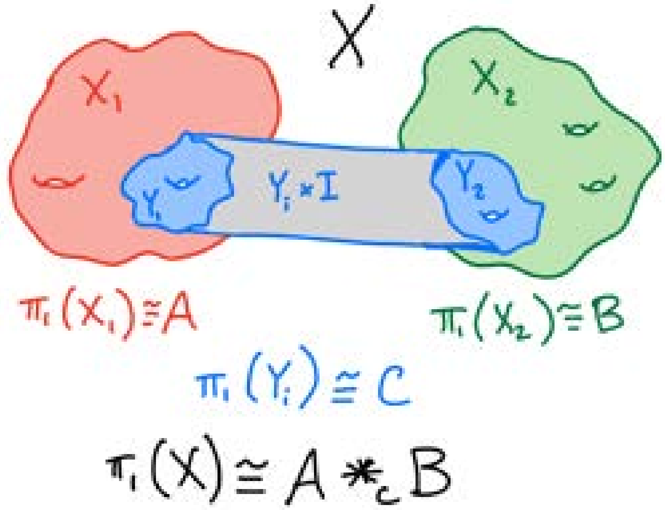

HNN Extensions

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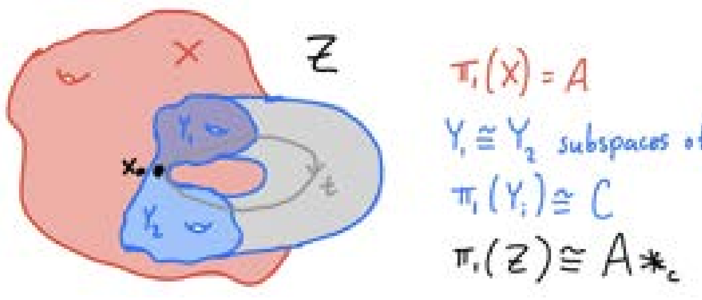

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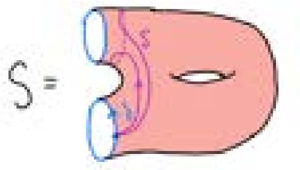

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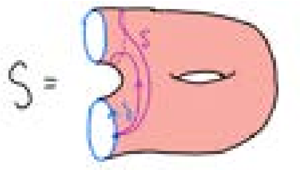

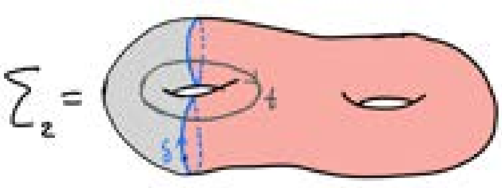

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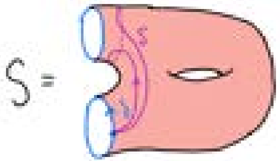
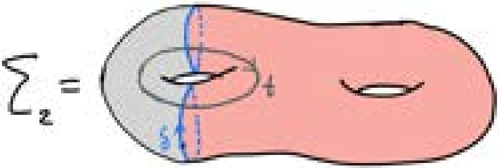
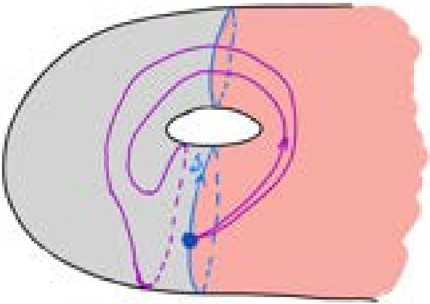




$$\pi_1(\Sigma_2) = \pi_1(S)*_{\mathbb{Z}}$$

Conjugation in $\pi_1(\Sigma_2)$

$\pi_1(\Sigma_2) = \pi_1(S) *_{\langle\delta\rangle} =$
$\langle \text{generators of } \pi_1(S),\ t \mid R_{\pi_1(S)}, t^{-1}\delta t = \delta' \rangle$



$\delta' = t^{-1}\delta t$

Application to Group Theory

Theorem (Grushko)

Let F be a finitely generated free group, $G = G_1 * G_2$ and let $\phi : F \to G$ be a surjective homomorphism. Then there are $H_1, H_2 < F$ such that $F = H_1 * H_2$ and $\phi(H_i) = G_i$.

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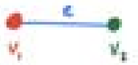

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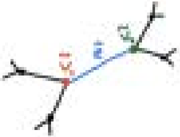
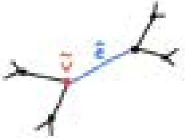

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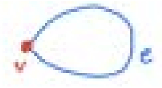

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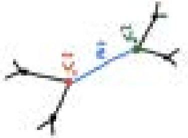

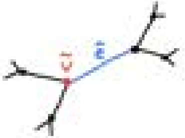

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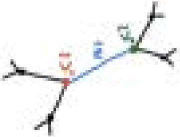

- Consider edge and vertex stabilizers. $G_{\tilde{e}} \hookrightarrow G_{\tilde{v}_i}$
- $G \cong G_{\tilde{v}_1} *_{G_{\tilde{e}}} G_{\tilde{v}_2} \qquad G \cong G_{\tilde{v}} *_{G_{\tilde{e}}}$

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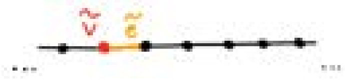

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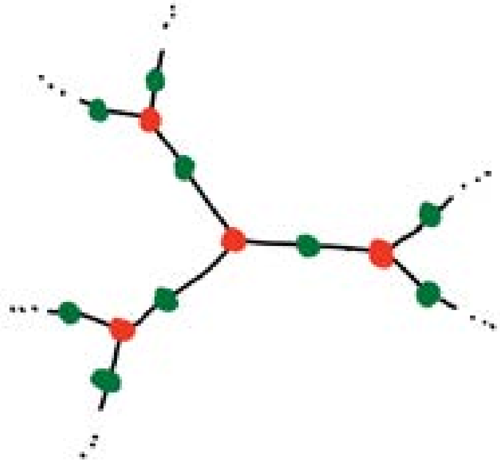
- $\mathbb{Z} \cong \{1\} *_{\{1\}}$

Example: $G = \mathbb{Z}_3 * \mathbb{Z}_2$

- $G = \mathbb{Z}_3 * \mathbb{Z}_2 = \langle a \mid a^3 \rangle * \langle b \mid b^2 \rangle$ acts on T



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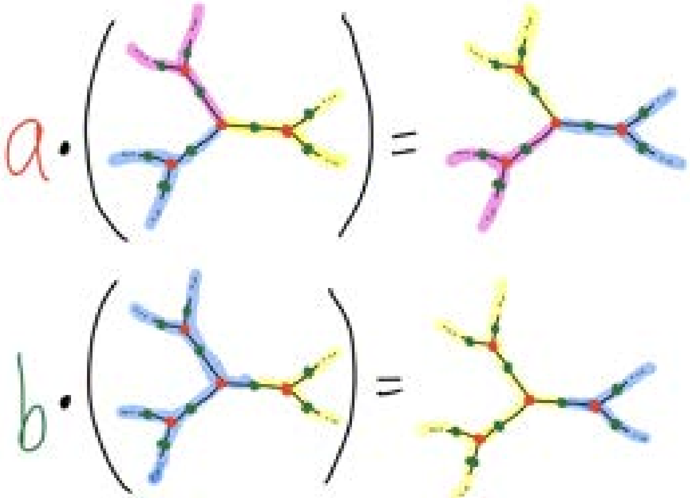

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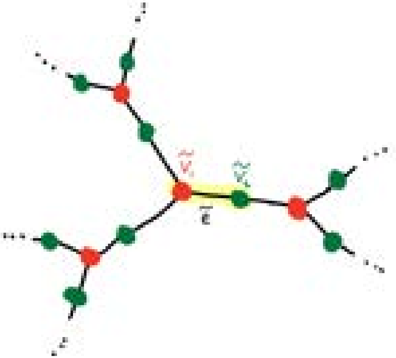

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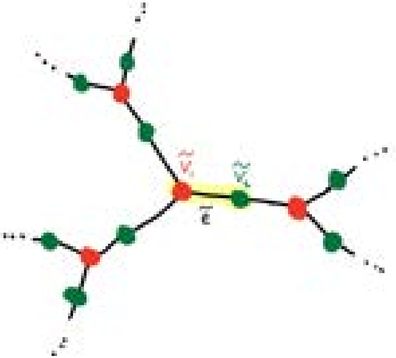

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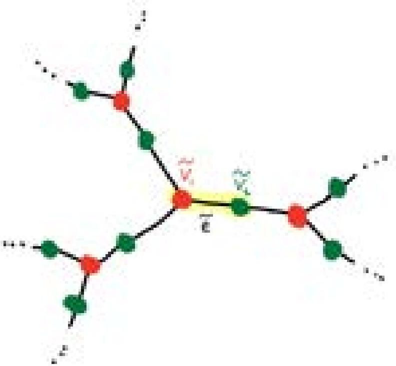
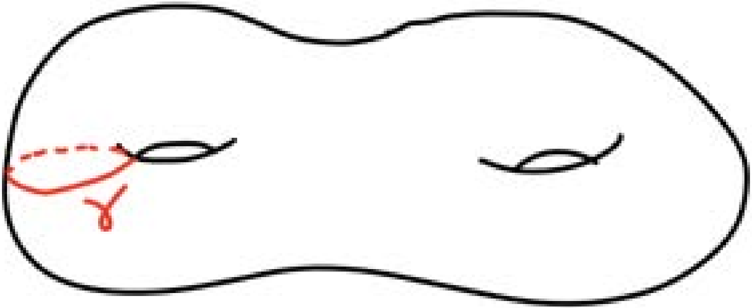


- $G_{\tilde{v}_1} = \langle a \rangle \cong \mathbb{Z}_3$, $G_{\tilde{v}_2} = \langle b \rangle \cong \mathbb{Z}_2$, $G_{\tilde{e}} = \{1\}$
- Recover $G \cong \mathbb{Z}_3 * \mathbb{Z}_2$

Example: Surface Group $G = \pi_1(\Sigma_2)$



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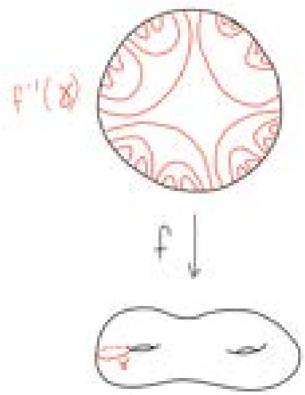

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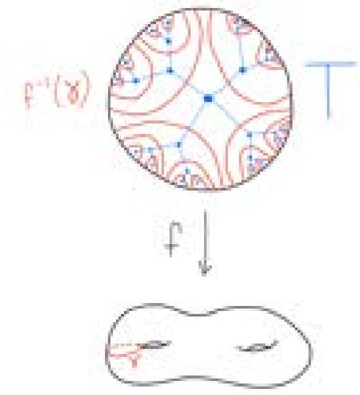

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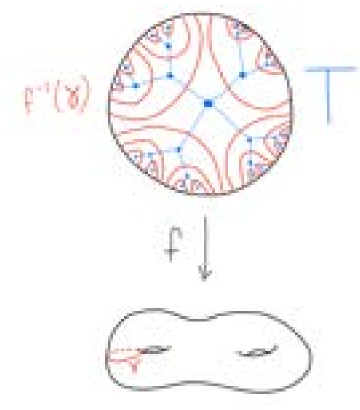

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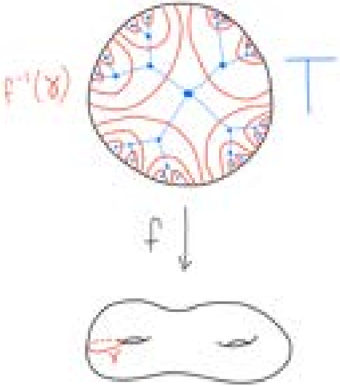

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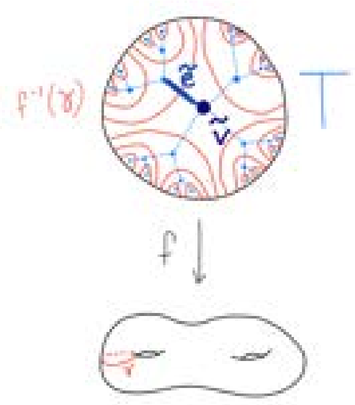

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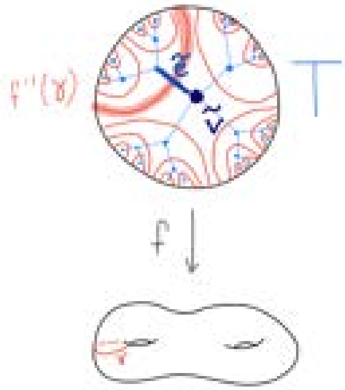

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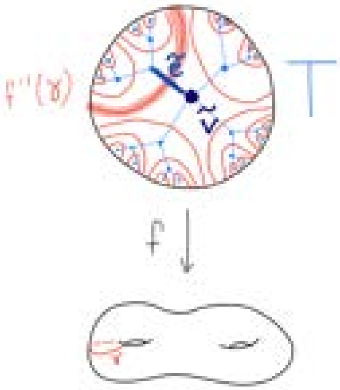

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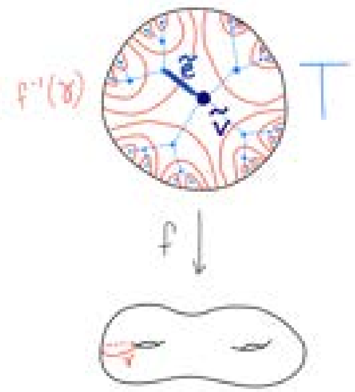

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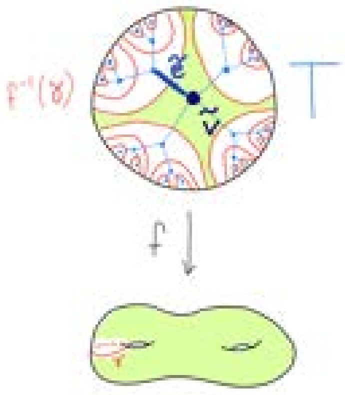

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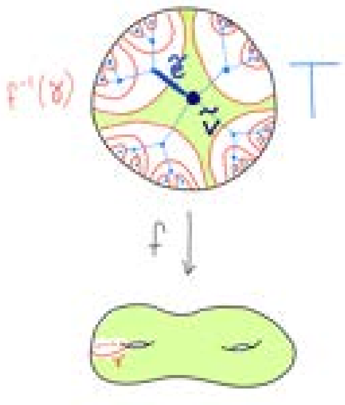

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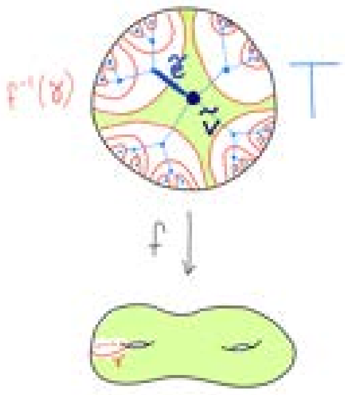

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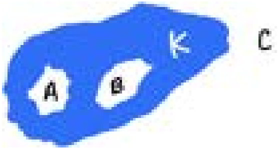
- $e(\mathbb{R}^n) = 1, \; n \geq 2$



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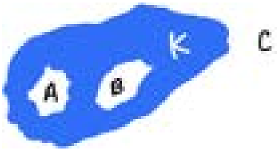
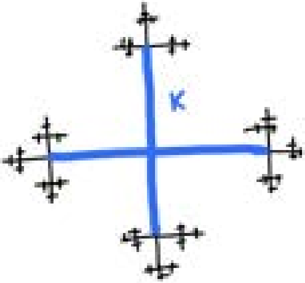


- X = infinite 4-valent tree, $e(X) = \infty$



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Ends of Groups

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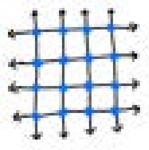

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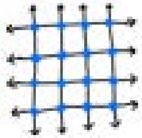
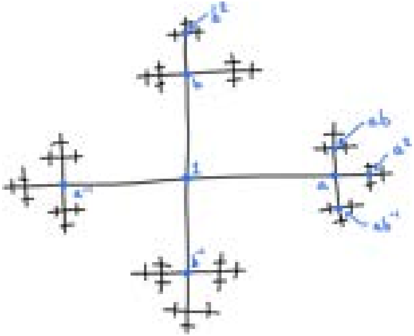


- $e(F_2) = \infty$ because $Cay(F_2, \{a, b\}) =$ infinite 4-valent tree



Universal Covers of Compact Simplicial Complexes

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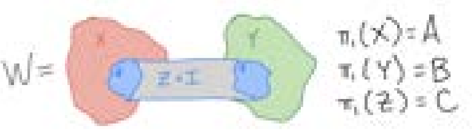

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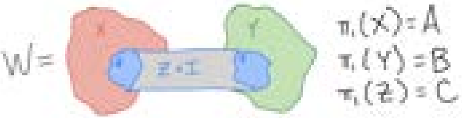

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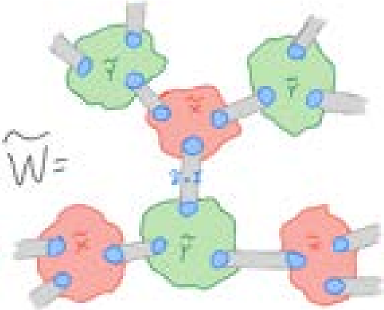

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- Induction on $rank(G)$ $\implies$ G_i free $\implies$ G free.

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