Exercises on D-modules

1. Let f be a homogeneous polynomial in $R := \mathbb{C}[x_1, \ldots, x_n]$. Consider the Euler operator

$$E := x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}.$$

   Verify that $E(f) = (\deg f)f$.

2. Let R be a commutative ring; recall that D_R^k denotes the differential operators on R of order up to k. Show that

$$D_R^k \circ D_R^l \subseteq D_R^{k+l}.$$

   It follows that $D_R := \bigcup_{k \geqslant 0} D_R^k$ is a ring!

3. Let $R := \mathbb{C}[x]$. Express the following elements of $D_{R|\mathbb{C}}$ in terms of the PBW basis:

   (a) $\partial^2 \circ x$

   (b) $\partial \circ f$, where $f \in R$

   (c) $\partial^2 \circ f$, where $f \in R$

4. Let $D_{R|\mathbb{C}}$ be the Weyl algebra, where $R := \mathbb{C}[x_1, \ldots, x_n]$. Show that the center of $D_{R|\mathbb{C}}$ is $\mathbb{C}$ as follows:

   (a) If $P \in D_{R|\mathbb{C}}$ is central, then it is an R-linear operator, and hence belongs to $\operatorname{Hom}_R(R, R) \cong R$.

   (b) For a polynomial $P \in R$, one has $[\partial_i,\ P] = \partial P / \partial x_i$.

   (c) Conclude that the center of $D_{R|\mathbb{C}}$ is $\mathbb{C}$.

5. Let $D_{R|\mathbb{C}}$ be the Weyl algebra, where $R := \mathbb{C}[x_1, \ldots, x_n]$. Let P be a nonzero element on $D_{R|\mathbb{C}}$.

   (a) If ∂_i occurs in P when P is expressed in terms of the PBW-basis, prove that $[P,\ x_i] \neq 0$.

   (b) If x_i occurs in P when P is expressed in terms of the PBW-basis, prove that $[P,\ \partial_i] \neq 0$.

   (c) Conclude (yes, once again!) that the center of $D_{R|\mathbb{C}}$ is $\mathbb{C}$.

6. Let f be an element of $\mathbb{C}[x]$. Prove that in $D_{R|\mathbb{C}}$ one has

$$\frac{\partial^k}{\partial x^k} \circ f \;=\; \sum_{i+j=k} \binom{k}{i} \left(\frac{\partial^i f}{\partial x^i} \right) \frac{\partial^j}{\partial x^j}.$$

7. Let $\mathscr{F}_\bullet$ denote the Bernstein filtration on the Weyl algebra $D_{R|\mathbb{C}}$. Prove that

$$[\mathscr{F}_i,\ \mathscr{F}_j] \subseteq \mathscr{F}_{i+j-2}.$$

8. Let $\mathscr{F}_\bullet$ denote the Bernstein filtration on the Weyl algebra $D_{R|\mathbb{C}}$. Take M to be $D_{R|\mathbb{C}}$ and define $\mathscr{G}_\bullet$ on M by $\mathscr{G}_t := M$ for all $t \geqslant 0$. Is $\operatorname{gr} M$ finitely generated over $\operatorname{gr} D_{R|\mathbb{C}}$?

9. Let $R := \mathbb{C}[x]$ and let $\mathscr{F}_\bullet$ denote the Bernstein filtration on $D_{R|\mathbb{C}}$. Consider the induced filtration on $R_x = R[1/x]$; specify a basis for

$$\mathscr{F}_t \cdot \frac{1}{x} \qquad \text{for each } t \geqslant 0.$$

   Use this to compute the multiplicity of R_x as a $D_{R|\mathbb{C}}$-module.

10. Let $R := \mathbb{C}[x]$. Fix $\lambda \in \mathbb{C}$, and consider the natural action of $D_{R|\mathbb{C}}$ on

$$M := \bigoplus_{i \in \mathbb{Z}} \mathbb{C}\, x^{\lambda+i}.$$

   (a) Compute $e(M)$, i.e., the multiplicity of M.

   (b) Prove that M is a simple $D_{R|\mathbb{C}}$-module if and only if $\lambda \notin \mathbb{Z}$.

11. Let $R := \mathbb{C}[x_1, \ldots, x_n]$. For $k \leqslant n$, determine the multiplicity of $R_{x_1 \cdots x_k}$ as a $D_{R|\mathbb{C}}$-module.

12. (Nuking a mosquito) Using the above, and the Čech complex $\check{C}^\bullet(x_1,\dots,x_n;R)$, prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^k \;=\; (-1)^n.$$

With conventional weapons, one could set $x := 2$ in the binomial expansion of $(1-x)^n$.

13. Prove that every holonomic module M over the Weyl algebra $D := D_{R|\mathbb{C}}$ is cyclic as follows:

    (a) Recall that $\ell(M)$ is finite; by induction, reduce to the case $M = Du + Dv$, where Dv is simple.

    (b) Since Du has finite length, there exists a nonzero P in D with $Pu = 0$.

    (c) Since $DPD = D$, one has $DPDv \neq 0$, so there exists $Q \in D$ with $PQv \neq 0$.

    (d) Show that $u + Qv$ generates M.

14. Let $R := \mathbb{C}[x]$ and set $D := D_{R|\mathbb{C}}$. Construct an isomorphism of left D-modules

$$D/Dx^2 \xrightarrow{\;\cong\;} D/Dx \oplus D/Dx.$$

If you are having fun, go for

$$D/Dx^3 \xrightarrow{\;\cong\;} D/Dx \oplus D/Dx \oplus D/Dx.$$

15. Consider a $2\times n$ matrix of indeterminates

$$Z := \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix},$$

and the $\mathbb{C}$-linear action of $G := \mathrm{SL}_2(\mathbb{C})$ on the polynomial ring $R := \mathbb{C}[Z]$, where $M \in G$ acts as

$$M\colon Z \longmapsto MZ.$$

The goal is to show that the invariant ring R^G is $S := \mathbb{C}[\Delta_{ij} : 1 \leqslant i < j \leqslant n]$, where $\Delta_{ij} := x_i y_j - x_j y_i$. Set

$$E_{ij} := x_i\frac{\partial}{\partial x_j} + y_i\frac{\partial}{\partial y_j} \qquad \text{and} \qquad D_{ij} := \det\begin{bmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{bmatrix}.$$

    (a) If $n = 1$, prove that $R^G = \mathbb{C}$.

    (b) Show that each E_{ij} acts on S.

    (c) Show that $E_{ij}\circ g \;=\; g\circ E_{ij}$ for each $g \in G$.

    (d) Show that each E_{ij} acts on R^G.

    (e) Show that each D_{ij} acts on R^G.

    (f) Prove Capelli's identity:

$$\det\begin{bmatrix} E_{ii}+1 & E_{ij} \\ E_{ji} & E_{jj} \end{bmatrix} \;=\; \det\begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix} \circ \det\begin{bmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{bmatrix},$$

    for $i \neq j$, where determinants are read left to right; in other words prove that

$$(E_{ii}+1)E_{jj} - E_{ji}E_{ij} \;=\; \Delta_{ij}D_{ij}.$$

    (g) Take the $\mathbb{N}^n$-grading on R with $\deg x_i = \deg y_i = e_i$, the i-th basis vector; show that R^G inherits a grading.

    (h) Prove that $R^G = S$ as follows: if not, choose a homogeneous f in $R^G \setminus S$ of degree $(d_1,\dots,d_n)$ such that $\sum d_i$ is minimal, and that, amongst such f, the entry d_1 maximal. Then $d_j \neq 0$ for some $j \neq 1$ by (a). Consider

$$(E_{11}+1)E_{jj}(f) \;=\; E_{j1}E_{1j}(f) + \Delta_{1j}D_{1j}(f).$$

16. Let $D_{R|\mathbb{C}}$ be the Weyl algebra, where $R := \mathbb{C}[x_1,\dots,x_n]$. Suppose M is a $D_{R|\mathbb{C}}$-module with a filtration $\mathscr{G}_\bullet$ for which there exist c, m such that

$$\operatorname{rank}_{\mathbb{C}}\mathscr{G}_t \;\leqslant\; ct^m \qquad \text{for all } t \gg 0.$$

Does M need to be finitely generated? (We saw that this is true if $m = n$.)

Hint: Take $n = 1$ and consider the $D_{R|\mathbb{C}}$-module

$$M := R \oplus R \oplus R \oplus \cdots$$

with the filtration

$$\mathscr{G}_t := [R]_{\leqslant t-1} \oplus [R]_{\leqslant t-2} \oplus [R]_{\leqslant t-3} \oplus \cdots.$$

17. (Symmetry of the Weyl algebra) Recall that for A a ring, the *opposite ring* A^{op} consists of A as an abelian group, with multiplication in "reverse order." More precisely,

$$A^{\mathrm{op}} := \{a^{\mathrm{op}} \mid a \in A\},$$

with $a^{\mathrm{op}} + b^{\mathrm{op}} = (a+b)^{\mathrm{op}}$, and $a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$. Let $D_{R|\mathbb{C}}$ be the Weyl algebra, where $R := \mathbb{C}[x_1, \dots, x_n]$.

   (a) Show that the $\mathbb{C}$-algebra map with $x_i \longmapsto x_i^{\mathrm{op}}$ and $\partial_i \longmapsto -\partial_i^{\mathrm{op}}$ gives an isomorphism $D_{R|\mathbb{C}} \longrightarrow D_{R|\mathbb{C}}^{\mathrm{op}}$.

   (b) Note that right $D_{R|\mathbb{C}}$-modules correspond to left modules over $D_{R|\mathbb{C}}^{\mathrm{op}}$. Using the fact that $D_{R|\mathbb{C}}$ is left noetherian, conclude that the ring $D_{R|\mathbb{C}}$ is also right Noetherian.

18. Set $R := \mathbb{F}_p[w,x,y,z]$ and $f := wx - yz$. Construct a differential operator $P \in D_{R|\mathbb{F}_p}$ such that

$$P(1/f) \;=\; 1/f^p.$$

19. Let $R := \mathbb{F}_p[x]$. Recall that $D_t := \dfrac{1}{t!}\dfrac{\partial^t}{\partial x^t}$ for $t \geqslant 1$.

   (a) Prove that $D_1^p = 0$.

   (b) Prove that $[D_q,\, x^q] = 1$ for each integer $q = p^e$.
   Hint: For q as above, and $m \in \mathbb{N}$, a theorem of Lucas implies that

$$\binom{m+q}{q} - \binom{m}{q} \;\equiv\; 1 \mod p.$$

20. For R and D_t as above. Prove that D_1 belongs to the $\mathbb{F}_p$-algebra generated by x and D_{p-1}.

21. (An application of differential operators to computing *F-thresholds*) Let f be a homogeneous cubic polynomial in $\mathbb{F}_p[x,y,z]$ for which the Jacobian ideal $J := (\partial f/\partial x,\ \partial f/\partial y,\ \partial f/\partial z)$ is primary to the homogeneous maximal ideal $\mathfrak{m} := (x,y,z)$. The goal is to show that

$$f^{p-2} \;\notin\; \mathfrak{m}^{[p]} := (x^p,\ y^p,\ z^p).$$

   (a) Let k be least such that $f^k \in \mathfrak{m}^{[p]}$. If $k < p$, show that $f^{k-1}J \subseteq \mathfrak{m}^{[p]}$.

   (b) Prove that $\mathfrak{m}^4 \subseteq J$.

   (c) Prove that $(\mathfrak{m}^{[p]} : \mathfrak{m}^4) \;=\; (\mathfrak{m}^{[p]} + \mathfrak{m}^{3p-6})$.

   (d) Conclude that $f^{k-1} \;\in\; (\mathfrak{m}^{[p]} + \mathfrak{m}^{3p-6})$, and hence that $\deg f^{k-1} \geqslant 3p-6$.

   (e) Conclude that $k \geqslant p-1$.

22. For p a prime integer, set W to be $\mathbb{F}_p\langle x,y\rangle / \langle [x,\ y] - 1\rangle$.

   (a) Prove that x^p is in the center of W.

   (b) Prove that R is not a simple ring, i.e., find a two-sided proper ideal.

23. The goal is to compute the center of the Weyl algebra in positive characteristic; let $R := \mathbb{F}_p[x_1, \dots, x_n]$ and consider the Weyl algebra

$$W := \mathbb{F}_p\langle x_1, \dots, x_n,\ \partial_1, \dots, \partial_n\rangle / \big\langle [x_i, x_j],\ \ [\partial_i, \partial_j],\ \ [\partial_i, x_j] - \delta_{ij} \big\rangle.$$

   (a) Show that there is an $\mathbb{F}_p$-algebra homomorphism $W \longrightarrow D_{R|\mathbb{F}_p}$. Show that it fails to be injective, and also fails to be surjective. Characterize the image in terms of the level filtration $D^{(e)} := \mathrm{Hom}_{R^{p^e}}(R,R)$.

   (b) Show that each x_i^p and ∂_i^p is in $Z(W)$, i.e., the center of W.

   (c) If $A \longrightarrow B$ is a ring homomorphism, show that $Z(A)$ need not map to $Z(B)$. However, show that $Z(A)$ must map to $Z(B)$ when $A \longrightarrow B$ is surjective. Using this, and your answer to (a), show that

$$Z(W) = k[x_1^p, \dots, x_n^p,\ \partial_1^p, \dots, \partial_n^p].$$

24. Set $R := \mathbb{Z}[x_1,x_2,x_3,\ y_1,y_2,y_3]/(x_1y_1+x_2y_2+x_3y_3)$ and $\mathfrak{a} := (x_1,x_2,x_3)R$. The goal is to show that $H^3_{\mathfrak{a}}(R)$ has infinitely many associated primes. Let p be an arbitrary prime integer; consider the cohomology class

$$\eta_p := \left[\frac{(x_1y_1)^p+(x_2y_2)^p+(x_3y_3)^p}{p(x_1x_2x_3)^p}\right] \quad \text{in} \quad H^3_{\mathfrak{a}}(R) \;=\; \frac{R_{x_1x_2x_3}}{R_{x_1x_2}+R_{x_1x_3}+R_{x_2x_3}}.$$

(a) Check that the fraction $\big((x_1y_1)^p+(x_2y_2)^p+(x_3y_3)^p\big)/p$ is indeed an element of R.

(b) Verify that $p\eta_p = 0$.

Prove that η_p is nonzero as follows: if $\eta_p = 0$, then there exists an integer k and elements c_i in R with

$$\frac{(x_1y_1)^p+(x_2y_2)^p+(x_3y_3)^p}{p}(x_1x_2x_3)^k \;=\; c_1x_1^{p+k}+c_2x_2^{p+k}+c_3x_3^{p+k}. \tag{1}$$

Consider the $\mathbb{N}^3$-grading on R with $\deg x_i = e_i$ and $\deg y_i = -e_i$, where e_i is the i-th basis vector.

(c) Without loss of generality, the c_i are homogeneous; determine the degree of each c_i.

(d) Conclude that c_1 is a scalar multiple of $y_1^p x_2^k x_3^k$, and draw similar conclusions for c_2 and c_3.

(e) Rewrite equation (1) using these observations; divide through by $(x_1x_2x_3)^k$, then specialize each $y_i \longrightarrow 1$, and $x_3 \longmapsto -(x_1+x_2)$, to obtain

$$\frac{x_1^p+x_2^p+(-x_1-x_2)^p}{p} \;\in\; \big(p,\ x_1^p,\ x_2^p\big)\mathbb{Z}[x_1,x_2].$$

(f) Prove that the above is false, so as to obtain a contradiction.

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