Complex Analysis Outline 2

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In this chapter, we discussed topology in $\mathbb{C}$.

1 Metric Spaces

Definition 1.1. A metric space satisfies the following conditions:

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y.x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Definition 1.2. A set S is open in metric space (X, d) if $\forall x \in S, \exists \varepsilon > 0$, such that $B(x, \varepsilon) \subset S$

Proposition 1.1. The union of open sets is open; the intersection of open sets is open. (Proof is ignored)

Definition 1.3. A set is closed if its complement is open.

Proposition 1.2. The union of closed sets is closed; the intersection of closed sets is closed, $\mathbb{R}$, and $\emptyset$ are both open and closed. (Proof is ignored)

And in the later section, we defined closure, interior, and boundary of a set.

Definition 1.4. A set A is dense if $\bar{A} = X$. (The closure of A equals to the metric space).

2 Connectedness

Definition 2.1. A metric space (X, d) if the only subsets of X which are both open and closed are the empty set or X itself.

Proposition 2.1. A set $X \subset \mathbb{R}$ is connected iff X is an interval. If $X \subset \mathbb{R}^n$ is connected iff X has a polygon connecting any two points in X.

Definition 2.2. A subset is a component of the metric space if it is a maximal connected subset in X

In the later section, we explored the property of connected sets.

3 Completeness

Definition 3.1. A metric space is complete iff all Cauchy sequences are convergent sequences

Proposition 3.1. $\mathbb{C}$ is complete.

Theorem 3.1. (Cantor's Intersection Theorem) A metric space is complete iff for any sequence of non-empty closed sets U_n with $U_1 \supset U_2 \supset ...$ and diam $U_n \to 0$, there is a point in the intersection of the sets.

Remark 3.1.1. This theorem could be easily proved by compactness as well.

In the rest of the section we talked about sequences and closed sets.

4 Compactness

Definition 4.1. A set U is compact if it satisfy one of the 3 conditions:

1. It is closed and bounded
2. it satisfies the Bolzano-Weiestrauss Property
3. it satisfies the Heine-Borel Property

In the rest of the section we explored the property of Compactness

5 Continuity and Uniform convergence

In this section, we explored continuity combined with topological properties. One of the curious discovery is that Uniform convergence is equivalent to Lipschitz.