CLASS NUMBER FOMULAE

TING GONG

Contents

1. Abstract

In this paper, we are going to follow a text written by Jarvis, to explore the concept of class numbers and class groups; moreover, we are going to find ways to calculate the class number over number fields by deriving a class number formula through Analytic Number Theory, with Dirichlet Unit Theorem. We will first explain some geometric techniques in order to prove the finiteness of class numbers and Dirichlet Unit Theorem, and then we will use Analytic Number Theory to derive the formula. This formula is very useful in computing the class number of a specific number field, and decide if the number field is a unique factorization domain by analyzing the class group of the number field.

2. Preliminaries

Definition 2.1. An *ideal* I in a commutative ring R is defined with the following properties:

(i) $0_R \in I$

(ii) if i and $j \in I$, then $i - j \in I$

(iii) if $i \in I$, $c \in \mathbb{R}$, then $ci \in I$.

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An ideal with only one generator is called a *principal ideal*.

Definition 2.2. A field K is a *number field* if it is a finite extension of $\mathbb{Q}$.

Definition 2.3. Let K be a number field, then $\mathbb{Z}_K$ is the *ring of integers* of K, with $\mathbb{Z}_K = \{\alpha \in K | \alpha \text{ is an algebraic integer}\}$

Definition 2.4. A *fractional ideal* of $\mathbb{Z}_K$ is a subset of K with the form $\frac{1}{\gamma}\mathfrak{c}$, with $\mathfrak{c}$ an ideal in $\mathbb{Z}_K$ and γ a non-zero element of $\mathbb{Z}_K$. The fractional ideal is principal if $\mathfrak{c}$ is principal.

Definition 2.5. Let R be an integral domain, then R is a *principal ideal domain* if every ideal is principal.

Definition 2.6. Let R be in a ring, and $u \in R$. If there exists $v \in R$ such that $uv = 1$, then u is a unit in R

Definition 2.7. Let $p \in R$. Then p is *irreducible* if
(i) p is not a unit.
(ii) whenever $p = ab$, then either a or b is a unit.

Definition 2.8. A ring R is a *unique factorization domain* if it is an integral domain in which every element $a \in R$ can be written as $a = up_1 \ldots p_n$, where u is a unit and each p_i irreducible.

Fact 2.9. *Let $\phi : R \to S$ be a ring homomorphism. Then there is an isomorphism*

$$R/ker\phi \cong im\phi$$

Definition 2.10. An *nth roots of unity* is a number $\zeta \in \mathbb{C}$ such that $\zeta^n = 1$.

Definition 2.11. After choosing a basis for K a number field, represent $a \in K$ as a matrix. Thus, we define *norm* as the determinant of a, denoted by $N_{K/\mathbb{Q}}(a)$.

3. Calculate Class Number through Algebra

3.1. **What is a Class Group and Class Number.** In order to explain what a class group is and how it works, we need several facts. [Thm. 4.31, 5.30, 5.32]{**Jar14**}

Fact 3.1. *A principal ideal domain (PID) is a unique factorization domain (UFD). If we do a contrapositive, we will see this fact as: If a domain doesn't have unique factorization, then there are some ideals that are not principal.*

Fact 3.2. *Ideals in a ring of integers of number field can be uniquely factorized into prime ideals. This implies that we could use fractional ideals to represent ideals in the ring of integers.*

Fact 3.3. *Fractional ideals and principal fractional ideals form an Abelian group.*

Definition 3.4. Let K be a number field, $\mathfrak{F}_K$ be the group of fractional ideals, and $\mathfrak{PF}_K$ be the group of principal fractional ideals. Then we define the quotient group $C_K = \dfrac{\mathfrak{F}_K}{\mathfrak{PF}_K}$ to be a *class group* of K. And we call the number of elements in this group, h_K, the *class number.*

Remark 3.5. From the definition, we observe that when $h_K = 1$, the class group C_K is trivial, meaning that the domain is a unique factorization domain; otherwise, it will not be a unique factorization domain.

3.2. **Class Numbers on Imaginary Number Fields.** In this section, we first introduce some preliminary concept and theorems, before we calculate class numbers in imaginary number fields.

Definition 3.6. A *quadratic form in n variables* is a homogeneous polynomial of degree 2, i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

Notice that this could be written as a product of vectors and matrices: Let $v = (x_1, ..., x_n)^t$ and $A = (a_{ij})$, then it could be written as v^tAv.

Definition 3.7. A *binary quadratic form* is a quadratic form in 2 variables, thus can be written as

$$f(x, y) = ax^2 + bxy + cy^2$$

this can be written as (a, b, c), and it has discriminant $b^2 - 4ac$.

Definition 3.8. A quadratic form is *positive definite* if $f(x, y) \geq 0$, for all $x, y \in \mathbb{R}$, and $f(x, y) = 0$ iff $x = y = 0$. Notice that this is equivalent to discriminant $b^2 - 4ac < 0$.

Definition 3.9. Quadratic forms $f(x, y)$ is *equivalent* to $g(x, y)$ if there exists $p, q, r, s \in \mathbb{Z}$, such that $ps - qr = 1$ and one can map $f(x, y)$ to $g(x, y)$ or other way round by $(x, y) \mapsto (px + qy, rx + sy)$. And notice that p, q, r, s forms a matrix, the mapping denotes a linear transformation, and the matrix in $GL_2(\mathbb{Z})$. Similarly, $f(x, y)$ is *properly equivalent* to $g(x, y)$ if p, q, r, s forms a matrix, the mapping denotes a linear transformation, and the matrix in $SL_2(\mathbb{Z})$

Definition 3.10. A form (a, b, c) is *reduced* if $-a < b \leq a < c$ or $0 \leq b \leq a = c$.

Remark 3.11. One could prove that every positive definite binary quadratic form is properly equivalent to a unique reduced form. The reason this is introduced is that every ideal in a ring of integers has a corresponding reduced quadratic form. Therefore, we classify all the ideals having the same reduced quadratic form into a equivalence class. Therefore, we want to show that there is a bijective relation between this equivalence class and ideal classes; so that we could draw a solution about class group and class number through studying quadratic forms.

Now, we could start proving how to compute class numbers. But since we know that the rings of integers of different number fields have different forms, we need to distinguish $\mathbb{Z}[\sqrt{d}]$ of whether $d \equiv 2, 3 \mod 4$ or $d \equiv 1 \mod 4$. We assume $d \equiv 2, 3 \mod 4$ for now.

Lemma 3.12. *Let $\mathfrak{a}$ be an ideal in the ring of integer $\mathbb{Z}_K$. Then there exists positive integers a, b, c such that*

$$\mathfrak{a} = a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z}$$

with $c|a$ and $c|b$

Proof. Take a to be the minimal integer in $\mathfrak{a}$, c as small as possible. We claim that in this setting, we could represent $\mathfrak{a}$. We need to check three properties:

(i) We know that $a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z} \subseteq \mathfrak{a}$, we only need to prove the other direction. Take $x + y\sqrt{d} \in \mathfrak{a}$,and $\exists m$ such that $x + y\sqrt{d} - m(b + c\sqrt{d}) = (x-mb)+(y-mc)\sqrt{d}$, where $0 \leq (y+mc) < c$. Since c is as small as possible, we know that $y + mc = 0$ or we find a smaller integer which contradicts our assumption. Now we discuss $(x - mb)$. Since a is minimal, we know that $(x - mb) \equiv 0 \mod a$, or we find a smaller integer in the ideal. Thus, we know that $\mathfrak{a} = a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z}$

(ii) Now we do the same trick. Since $a \in \mathfrak{a}$, we conclude that $a\sqrt{d} \in \mathfrak{a}$. Therefore, $\exists t, a\sqrt{d} = t(b + c\sqrt{d}) + qa$ such that $0 \leq ct - a < c$. If $ct - a = 0$, $c|a$, otherwise we contradict the minimality of c

(iii) By the same reasoning as in (ii), $b\sqrt{d} + cd \in \mathfrak{a}$. Therefore, $\exists t$ such that $0 \leq b - ct < c$. Then we could conclude that $c|b$. □

Corollary 3.13. *Assume we could write $\mathfrak{a} = a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z}$ in a ring of integer $\mathbb{Z}_K$. Then $N_{K/\mathbb{Q}}(\mathfrak{a}) = ac$*

Proof. The norm of an ideal denotes the cardinality of the ring of integer modulo the ideal. Therefore $N_{K/\mathbb{Q}}(\mathfrak{a}) = |\mathbb{Z}_K/\mathfrak{a}|$. But we know that a, c are minimal, the set $\mathbb{Z}_K/\mathfrak{a} = \{x + y\sqrt{d} | 0 \leq x < a, 0 \leq y < c\}$. Therefore, it is clear that there are ac elements. □

Proposition 3.14. *Assume $a, b, c \in \mathbb{Z}$, then the $\mathbb{Z}$-module $a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z}$ is a ideal in $\mathbb{Z}_K$ iff $c|a$, $c|b$, and $ac|c^2d - b^2$*

Proof. The difference between a $\mathbb{Z}$-module and an ideal is that whether we could multiply it by an element in the ring and still remain in the set. Therefore, take $x, y \in \mathbb{Z}$, $\alpha = ax + by + cy\sqrt{d}$. And we know that $\alpha\sqrt{d} \in \mathfrak{a}$. Thus, $\exists s, t \in \mathbb{Z}, \alpha\sqrt{d} = cyd + ax\sqrt{d} + by\sqrt{d} = as + bt + ct\sqrt{d}$. Therefore, we know that $t = \dfrac{ax + by}{c}$, which implies that $c|a$, $c|b$, then $\forall x, y \in \mathbb{Z}$. Also, we have $cyd = as + bt$; thus we have $s = \frac{cyd-bt}{a} = \frac{c^2yd-abx-b^2y}{ac}$. This is an integer iff $ac|c^2d - b^2$ □

Theorem 3.15. *Assume $\mathfrak{a} = a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z}$ is an ideal of $\mathbb{Z}_K$. Then*

$$\frac{N_{K/\mathbb{Q}}(ax + (b + c\sqrt{d})y)}{N_{K/\mathbb{Q}}(\mathfrak{a})}$$

is a quadratic form with integer coefficients and discriminant $4d$.

Proof. We first calculate

$$\begin{aligned} N_{K/\mathbb{Q}}(ax + (b + c\sqrt{d})y) &= (ax + by)^2 - dc^2y^2 \\ &= a^2x^2 + 2axby + b^2y^2 - dc^2y^2 \end{aligned}$$

Therefore, we know that our original equation

$$\begin{aligned} \frac{N_{K/\mathbb{Q}}(ax + (b + c\sqrt{d})y)}{N_{K/\mathbb{Q}}(\mathfrak{a})} &= \frac{a^2x^2 + 2axby + b^2y^2 - dc^2y^2}{ac} \\ &= \frac{a}{c}x^2 + \frac{2b}{c}xy + \frac{b^2 - dc^2}{ac}y^2 \end{aligned}$$

Thus, by the last proposition, we know that this is a quadratic form with integer coefficients. And the discriminant

$$D_K = \frac{4ab^2 - 4ab^2 + 4adc^2}{ac^2} = 4d$$

□

Now we find a mapping from the ideals to quadratic forms, that

$$\Phi(\mathfrak{a}) = \frac{N_{K/\mathbb{Q}}(ax + (b + c\sqrt{d})y)}{N_{K/\mathbb{Q}}(\mathfrak{a})}$$

Before we spend time examine the bijectivity, we need to check that properly equivalent quadratic forms lie in one class.

Lemma 3.16. *If z is in the upper-half complex plane, and $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Z})$, then $\dfrac{q + sz}{p + rz}$ is in the upper half complex plane iff $M \in SL_2(\mathbb{Z})$*

Proof. Now we simplify the fraction:

$$\frac{q + sz}{p + rz} = \frac{(q + sz)(p + r\bar{z})}{|p + rz|^2} = \frac{pq + qr\bar{z} + psz + sr|z|^2}{|p + rz|^2}$$

Thus, the imaginary part is $\dfrac{im(z)(ps - qr)}{|p + rz|^2}$, therefore, it is clear that both direction works in this case. □

Proposition 3.17. *If $\mathfrak{a}$ and $\mathfrak{b}$ are in the same ideal class, then $\Phi(\mathfrak{a})$ and $\Phi(\mathfrak{b})$ are properly equivalent.*

Proof. Since $\mathfrak{a}$ and $\mathfrak{b}$ are in the same ideal class, $\exists\theta = \alpha/\beta$, such that $\mathfrak{a} = \theta\mathfrak{b}$, thus, $\beta\mathfrak{a} = \alpha\mathfrak{b}$. Now assume that $\mathfrak{a} = \gamma\mathbb{Z} + \delta\mathbb{Z}$, then $\beta\mathfrak{a} = \beta\gamma\mathbb{Z} + \delta\beta\mathbb{Z}$. Thus, we know that

$$N_{K/\mathbb{Q}}(\langle\beta\rangle\mathfrak{a}) = |N_{K/\mathbb{Q}}(\beta)|N_{K/\mathbb{Q}}(\mathfrak{a})$$

and

$$N_{K/\mathbb{Q}}(\beta\gamma x + \delta\beta y) = N_{K/\mathbb{Q}}(\beta)N_{K/\mathbb{Q}}(\gamma x + \delta y)$$

Thus, we know that $\Phi(\beta\mathfrak{a}) = \Phi(\mathfrak{a})$. And we could deduce $\Phi(\alpha\mathfrak{b}) = \Phi(\mathfrak{b})$. Therefore, we know that $\Phi(\alpha\mathfrak{b}) = \Phi(\beta\mathfrak{a})$ □

We define $\Psi((a, b, c)) = a\mathbb{Z} + (\frac{b}{2} + \sqrt{d})\mathbb{Z}$ and claim that this is the inverse function of Φ. We check:

Proposition 3.18. *If (a, b, c) and (a', b', c') are properly equivalent, then $\Psi((a, b, c))$ and $\Psi((a', b', c'))$ lie in the same ideal class.*

Proof. Since there are only three types of proper equivalence, we check all the possibilities: $(a, b, c) \mapsto (a, b \pm 2a, c \pm b + a)$ and $(a, b, c) \mapsto (c, -b, a)$
$\Psi((a, b\pm 2a, c\pm b+a)) = a\mathbb{Z}+(\frac{b \pm 2a}{2}+\sqrt{d})\mathbb{Z} = a\mathbb{Z}+(\frac{b}{2}+\sqrt{d})\mathbb{Z} = \Psi((a, b, c))$
Since we know that $b^2 - 4ac = 4d$, we have $-a = \frac{b^2-4d}{4c}$

$$\begin{aligned}\frac{b + 2\sqrt{d}}{2c}\Psi((c, -b, a)) &= \frac{b + 2\sqrt{d}}{2c}(c\mathbb{Z} + (\frac{-b}{2} + \sqrt{d})\mathbb{Z}) \\ &= (\frac{b}{2} + \sqrt{d})\mathbb{Z} + (-a)\mathbb{Z} = \Psi((a, b, c))\end{aligned}$$

Therefore, we know that they are in the same ideal class. □

Theorem 3.19. *Φ and Ψ are inverse bijections to each other between the set of proper equivalence calsses of quadratic forms and the set of ideal classes in $\mathbb{Z}[\sqrt{d}]$.*

Proof. Now, it suffices to check $\Phi(\Psi(a, b, c)) = (a, b, c)$ and $\Psi(\Phi(\mathfrak{a})) = \mathfrak{a}$.
$\Psi(\Phi(\mathfrak{a})) = \Psi(\frac{a}{c}x^2 + \frac{2b}{c}xy + \frac{b^2 - dc^2}{ac}y^2) = \frac{1}{c}(a\mathbb{Z} + (b + c\sqrt{d})\mathbb{Z})$
Therefore, we know that $\Psi(\Phi(\mathfrak{a}))$ gives a equivalence.

$$\begin{aligned}\Phi(\Psi(a, b, c)) &= \Phi((a\mathbb{Z} + (\frac{b}{2} + \sqrt{d})\mathbb{Z})) \\ &= \frac{1}{a}(a^2x^2 - abxy - \frac{b^2 - 4d}{4}y^2) \\ &= ax^2 + bxy + cy^2\end{aligned}$$

And this concludes our proof, since it gives exactly (a, b, c). □

Remark 3.20. We could do the same proof to $d \equiv 1 \mod 4$, but we only need to replace $\sqrt{d}$ with $\frac{1+\sqrt{d}}{2}$

Theorem 3.21. *There are only finitely many reduced quadratic forms of discriminant D.*

Proof. Assume (a, b, c) to be reduced, with $0 \leq |b| \leq a \leq c$. Thus $0 \leq b^2 \leq ac$. Which yields $-4ac \leq D \leq 3ac$. Therefore, we have the range of ac, which is $\frac{-D}{4} \leq ac \leq \frac{-D}{3}$. Thus, $a^2 \leq ac \leq \frac{-D}{3}$. We found a to be bounded, and for each choice of a and b, we have only one c. This tells us that there are finitely many reduced form. □

Corollary 3.22. *The class group of an imaginary quadratic field is finite.*

Remark 3.23. We can find specific class numbers by using the bound in Theorem 2.24, and counting quadratic forms.

4. Finiteness of Class Number and Dirichlet's Unit Theorem

In this section, we will need some geometrical techniques. This will be introduced in the following sections.

4.1. Finiteness of Class Number.

Definition 4.1. Let V be an n-dimensional real vector space. A *lattice* in V is a subgroup in the form

$$\Gamma = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$$

where $\{v_1 ... v_m\}$ are linearly independent vectors in V. The lattice is *complete* if $m = n$

Definition 4.2. The *fundamental mesh or fundamental region* associated to Γ, Φ_Γ, is defined as

$$\Phi_\Gamma = \{\alpha_1 v_1 + \ldots + \alpha_m v_m | 0 \leq \alpha_i < 1\}$$

Definition 4.3. A subset $\Gamma \subset \mathbb{R}^n$ is discrete if for all radius $r \geq 0$, Γ contains only finitely many points at a radius at most r from 0.

Definition 4.4. A region $X \subset V$ is *centrally symmetric* if $x \in X$ implies $-x \in X$

Definition 4.5. A region $X \subset V$ is *convex* if $x, y \in X$, and $t \in [0, 1]$ then the line $\{(1-t)x + ty\} \subset X$

Now, we recognize three basic propositions, to which I will give a sketch for the proofs in order to introduce a theorem faster.

Proposition 4.6. *A subgroup $\Gamma \subset V$ is a lattice iff it is discrete*

Sketch. ($\Rightarrow$) We could define a continuous map: $\phi : a_1v_1 + ... + a_nv_n \mapsto (a_1, ...a_n)$. We could draw a closed ball with radius r in the preimage, which would be compact. Therefore, the image would also be compact, thus we could take M to be the maximum and claim that we have $a_i \leq M$.

($\Leftarrow$) We could let Γ span V_0 and take Γ_0 to be lattices in V_0, and we could prove that $q\Gamma = \Gamma_0$ by discussing the extra points besides Γ_0. □

Proposition 4.7. *A subgroup $\Gamma \subset V$ is complete iff $\exists$ a bounded $B_V \in V$ such that $\bigcup_{\gamma\in\Gamma}(B_V + \gamma)$.*

Sketch. ($\Rightarrow$) Take B_v to be Φ_Γ

($\Leftarrow$) B_V is bounded, then every point is inside a radius r. If Γ is not complete, and V_0 is the span, then V_0 is not V. Then there will be points out side of V_0 but inside V, which will lead to a contradiction. □

Proposition 4.8. *Assume Γ is a lattice in $\mathbb{R}^n$. If $v_i = (a_{i1}, ..., a_{in})$, then the volume $vol(\Gamma) = |\det(a_{ij})|$*

Sketch. By changing of coordinates when computing the integral:

$$\int_{\Phi_\Gamma} 1dx_1...dx_n$$

□

Theorem 4.9. *(Minkowski) Assume Γ is a complete lattice in V. Let X be a centrally symmetric convex subset of V. Suppose $vol(X) > 2^n vol(\Gamma)$, then X contains at least one non-zero lattice point.*

Proof. We prove this by contradiction. Assume there are no non-zero lattice points. Then it is clear that $(\frac{1}{2}X + \gamma_1) \cap (\frac{1}{2}X + \gamma_2) = \emptyset$, where γ_1 and γ_2 are distinct lattices (if not, then we can find $x_1, x_2 \in X$ such that $\gamma_1 - \gamma_2 = \frac{1}{2}x_2 - \frac{1}{2}x_1$). Then, we know that $\{\Phi_\Gamma \cap \frac{1}{2}X + \gamma\}_{\gamma\in\Gamma} = \emptyset$. But this is a subset of Φ_Γ. Thus $vol(\Gamma) \geq vol(\Phi_\Gamma \cap \{\frac{1}{2}X + \gamma\}_{\gamma\in\Gamma}) = vol(\frac{1}{2}X) = \frac{1}{2^n}vol(X)$, which is a contradiction. □

Definition 4.10. If σ: $K \hookrightarrow \mathbb{C}$, and $\sigma(K) \subset \mathbb{R}$, then σ is called a *real embedding*. Otherwise it is called a *complex embedding*. Its conjugate denoted by $\bar{\sigma}$ is defined as $\bar{\sigma}(k) = \overline{\sigma(k)}$.

Proposition 4.11. *Let $K_\mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ (Since $\mathbb{C} \cong \mathbb{R}^2$, we could understand this space as a real space with $(r_1 + 2r_2)$-dimensional space). And i be a mapping from $K \hookrightarrow K_\mathbb{R}$. $\Gamma = i(\mathbb{Z}_K)$ is a complete lattice in $K_\mathbb{R}$ and $vol(\Gamma) = |D_K|^{1/2}$.*

Proof. Assume $\Gamma = \mathbb{Z}i\omega_1 + ... + \mathbb{Z}i\omega_n \subset K_\mathbb{R}$. Let M be the matrix $(\tau_i\omega_j)$. Then, by definition, we know that $D_K = \triangle\{\omega_1, ..., \omega_n\} = \det(M)^2$. Then, by the reasoning in proposition 3.8, we know that $vol(\Gamma) = |det(\tau_i\omega_j)| = det(M) = |D_K|^{1/2}$ □

Definition 4.12. The *discriminant of ideal* $\mathfrak{a}$, if $\mathfrak{a} = \alpha_1\mathbb{Z} + ... + \alpha_n\mathbb{Z}$, is $D(\mathfrak{a}) = \triangle\{\alpha_1, ..., \alpha_n\} = \det(\tau_i\alpha_j)^2$, where τ are all embeddings of K into $\mathbb{C}$.

Corollary 4.13. *If* $\mathfrak{a}$ *is a non-zero ideal of* $\mathbb{Z}_K$*, then* $\Gamma = i(\mathfrak{a})$ *is a complete lattice in* $K_\mathbb{R}$*, with* $D(\mathfrak{a}) = N_{K/\mathbb{Q}}(\mathfrak{a})^2 D_K$*, and* Φ_r *has volume* $N_{K/\mathbb{Q}}(\mathfrak{a})|D_K|^{1/2}$

Proposition 4.14. *Let* Γ *be a lattice in* $K_\mathbb{R}$*, and let* $c_1, ..., c_{r_1}, C_1, ..., C_{r_2} \in \mathbb{R}_{>0}$ *satisfy* $c_1...c_{r_1}(C_1...C_{r_2})^2 > \left(\frac{2}{\pi}\right)^{r_2} vol(\Gamma)$*. Then there exists a non-zero* $v = (x_1, ..., x_{r_1}, z_1, ..., z_{r_2}) \in \Gamma$ *such that* $|x_j| < c_j$ *for all* $j = 1, ..., r_1$*, and* $|z_k| < C_k$ *for all* $k = 1, ..., r_2$*.*

Proof. In this proof, we want to invoke Minkowski's theorem. Let X be the set of all elements with $|x_j| < c_j$ for all $j = 1, ..., r_1$, and $|z_k| < C_k$ for all $k = 1, ..., r_2$. Then it is clear that X is centrally symmetric and convex. Then we have $vol_\mathbb{R}(X) > (2c_1)...(2c_{r_1})(\pi C_1^2)...(\pi C_{r_n}^2)$. Thus, we know that $vol(X) = 2^{r_2} vol_\mathbb{R}(X) > 2^{r_2}(2c_1)...(2c_{r_1})(\pi C_1^2)...(\pi C_{r_n}^2) > 2^{r_1+r_2}\pi^{r_2}\left(\frac{2}{\pi}\right)^{r_2} vol(\Gamma)$. Therefore, we finally get $vol(X) > 2^n vol(\Gamma)$. Thus we know v exists. □

Proposition 4.15. *Let* $\mathfrak{a}$ *be a non-zero integral ideal of* $\mathbb{Z}_K$*. Then there exists a non-zero* $\alpha \in \mathfrak{a}$ *such that* $|N_{K/\mathbb{Q}}(\alpha)| \leq \left(\frac{2}{\pi}\right)^{r_2} N_{K/\mathbb{Q}}(\mathfrak{a})|D_K|^{1/2}$

Proof. By Corollary 3.13, we take M, where

$$M > \left(\frac{2}{\pi}\right)^{r_2} N_{K/\mathbb{Q}}(\mathfrak{a})|D_K|^{1/2} = \left(\frac{2}{\pi}\right)^{r_2} vol(\alpha)$$

Therefore, by proposition 3.14, we could choose $c_1...c_{r_1}(C_1...C_{r_2})^2 = M$. Therefore, there is a non-zero element $\alpha \in \mathfrak{a}$, such that each of its coordinates is smaller than the embeddings. Therefore, we know that $N_{K/\mathbb{Q}}(\alpha) < M$. Since M can be infinitely close to the value, we know that we get the equation required. □

Theorem 4.16. *The class group* $C(K)$ *is finite.*

Proof. We take $\mathfrak{b} \in [\mathfrak{a}^{-1}]$, where $[\mathfrak{a}^{-1}]$ denotes the ideal class of $\mathfrak{a}^{-1}$, WLOG, we assume $\mathfrak{b} \in \mathbb{Z}_K$. Then by proposition 3.15, we have $\exists\beta$ such that $|N_{K/\mathbb{Q}}(\beta)| \leq \left(\frac{2}{\pi}\right)^{r_2} N_{K/\mathbb{Q}}(\mathfrak{b})|D_K|^{1/2}$. Then let $\mathfrak{c} = \langle\beta\rangle\mathfrak{b}^{-1} \in [\mathfrak{a}]$. Therefore, we have $N_{K/\mathbb{Q}}(\mathfrak{c}) = |N_{K/\mathbb{Q}}(\beta)|N_{K/\mathbb{Q}}(\mathfrak{b})^{-1} \leq \left(\frac{2}{\pi}\right)^{r_2}|D_K|^{1/2} = M$. Therefore, we know there are finitely ideals whose norm is within a bound. Thus there are only finitely many ideal classes. □

Now, let's find a better bound.

Lemma 4.17. *Let*

$X_t = \{(x_1, ..., x_{r_1}, z_1, ..., z_{r_2}) || x_1| + ... + |x_{r_1}| + 2|z_1| + ... + 2|z_{r_2}| < t\} \subset K_{\mathbb{R}}$

Then $vol(X_t) = 2^{r_1}\pi^{r_2}\frac{t^n}{n!}$

Proof. Since $\mathbb{C} \cong \mathbb{R}^2$, we could see each z_i as u_i, v_i. And by a former proposition, $vol(X) = 2^{r_2}vol_{\mathbb{R}}(X)$, we only need to calculate $vol_{\mathbb{R}}(X)$ by changing variables (u_i, v_i) to $(\frac{R_i}{2}\cos\theta_i, \frac{R_i}{2}\sin\theta_i)$, thus

$$\begin{aligned} vol_{\mathbb{R}}(X) &= \int_{X_t} 1dx_1...dx_{r_1}du_1dv_1...du_{r_2}dv_{r_2} \\ &= 2^{r_1}4^{-r_2}(2\pi)^{r_2}\int_{Y_t} R_1..R_{r_2}dx_1...dx_{r_1}dR_1...dR_{r_2} \\ &= 2^{r_1}4^{-r_2}(2\pi)^{r_2}\frac{t^n}{n!} \end{aligned}$$

Therefore, we have $vol(X_t) = 2^{r_1}\pi^{r_2}\frac{t^n}{n!}$ □

Theorem 4.18. *(Minkowski bound) Every ideal class of K contains an integral ideal $\mathfrak{c}$ of norm at most* $\frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}|D_K|^{1/2}$

Proof. Assume $\mathfrak{b} \in [\mathfrak{a}^{-1}]$, with $\mathfrak{a}$ an ideal class. We want to invoke Minkowski's theorem, so we choose a t, such that $2^{r_1}\pi^{r_2}\frac{t^n}{n!} > 2^n vol(\mathfrak{b})$. Since $n = r_1 + 2r_2$, by a proposition proved before, we pick

$$t^n > n!\left(\frac{4}{\pi}\right)^{r_2}|D_K|^{1/2}N_{K/\mathbb{Q}}(\mathfrak{b})$$

Then we know there exists a non-zero $\beta \in \mathfrak{b}$ with $i(\beta) \in X_t$. Moreover, we have Arithmetic Mean-Geometric Mean inequality giving

$$\begin{aligned} \left(\prod_\tau |\tau(\beta)|\right)^{\frac{1}{n}} &\leq \frac{1}{n}\left(\sum_\tau |\tau(\beta)|\right) \\ |N_{K/\mathbb{Q}}(\beta)| &\leq \left(\frac{t}{n}\right)^n \\ |N_{K/\mathbb{Q}}(\beta)| &< \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}N_{K/\mathbb{Q}}(\mathfrak{b})|D_K|^{1/2} \end{aligned}$$

Therefore, if $\mathfrak{c} = \langle\beta\rangle\mathfrak{b}^{-1} \in [\mathfrak{a}]$, then plug in our previous result into $N_{K/\mathbb{Q}}(\mathfrak{c}) = |N_{K/\mathbb{Q}}(\beta)|N_{K/\mathbb{Q}}(\mathfrak{b})^{-1}$, we get our result. □

4.2. **Dirichlet's Unit Theorem.**

Remark 4.19. We are going to introduce several mappings for future use:

$$l : (x_1, ..., x_{r_1}, z_1, ..., z_{r_2}) \mapsto (\log|x_1|, ..., \log|x_{r_1}|, \log|z_1|, ..., \log|z_{r_2}|)$$

and

$$\mathbb{Z}_K^\times = \{\epsilon \in \mathbb{Z}_K | N_{K/\mathbb{Q}}(\epsilon) = \pm 1\}$$
$$S = \{y \in K_\mathbb{R}^\times | N(y) = \pm 1\}$$

and we have

$$H = \{x \in \mathbb{R}^{r_1+r_2} | tr(x) = 0\}$$

Also,

$$\lambda : \mathbb{Z}_K^\times \xrightarrow{i} S \to\xrightarrow{l} H$$

And let $\Gamma = \lambda(\mathbb{Z}_K^\times)$.

Proposition 4.20. *The kernel of λ is $\mu(K)$, group of roots of unity in K.*

Proof. $\eta(K) \subseteq ker(\lambda)$ is clear. The embeddings clearly map the roots of unity to 0.

Now we prove the other direction. If $\epsilon \in ker(\lambda)$, then $|i(\epsilon)| = 1$. Therefore, it is a bounded region. And it is a lattice, thus it is discrete, thus it is finite. And since the kernel is closed under multiplication, we know every element has finite order. Thus it is a root of unity. □

Corollary 4.21. *Γ is a subgroup of H.*

Proposition 4.22. *Γ is a lattice in H.*

Proof. It suffices to prove that Γ is discrete. Thus, we want to show if $B(r, h) \subset H$, then $\Gamma \cap B(r, h)$ is finite. Consider $l^{-1}(\Gamma \cap B) = l^{-1}(\Gamma) \cap l^{-1}(B)$. Since $l^{-1}(\Gamma) = i(\mathbb{Z}_K^\times)$. We know $i(\mathbb{Z}_K^\times)$ is finite, $i(\mathbb{Z}_K^\times) \cap l^{-1}(B)$ is finite. And $l^{-1}(B)$ is bounded. Thus Γ is discrete. □

Proposition 4.23. *There is a bounded region $B_S \subset S$ such that*

$$S = \bigcup_{\epsilon \in \mathbb{Z}_K^\times} i(\epsilon) B_S$$

Proof. Consider the lattice $i(\mathbb{Z}_K^\times) \in K_\mathbb{R}$ of volume $|D_K|^{1/2}$. Then if we move the lattice by y, we have $yi(\mathbb{Z}_K^\times)$ also have volume $|D_K|^{1/2}$, because $N(y) = \pm 1$. Then we can appeal to proposition Prop. 3.14 to set up a X contains a non-zero point $x \in yi(\mathbb{Z}_K^\times)$, and thus we have $N(x) = N_{K/\mathbb{Q}}(\alpha)$, with $\alpha \in \mathbb{Z}_K^\times$. Therefore, we know that $N_{K/\mathbb{Q}}(\alpha)$ is bounded by an M from 3.14. Thus, there are only finitely many α, thus we construct a set $\{\alpha_1, ..., \alpha_N\}$. Thus, $\alpha = \epsilon^{-1}\alpha_k$. Therefore, we know that $y = xi(\alpha)^{-1} = xi(\alpha_k)^{-1}i(\epsilon)$. Therefore, we could take $B_S = \{s \in S | s \in Xi(\alpha_k)^{-1}\}$. □

Corollary 4.24. *Γ is a complete lattice in H.*

Proof. By last proposition, $S = \bigcup_{\epsilon \in \mathbb{Z}_K^\times} i(\epsilon) B_S$, take $B_H = l(B_S)$. We know that B_S is a translate of X. And $N(x) = \pm 1$, thus all the coordinates are bounded away from 0. Thus the logarithm is not a problem. Thus, B_H is bounded. Thus, we let $H = \bigcup_{\epsilon \in \mathbb{Z}_K^\times} (\lambda(\epsilon) + B_H) = \bigcup_{\gamma \in \Gamma} (\gamma + B_H)$. Therefore, by proposition 3.7, we know that Γ is complete. $\square$

Theorem 4.25. *(Dirichlet) $\exists \epsilon_1, ..., \epsilon_r$ such that all $\epsilon \in \mathbb{Z}_K^\times$ can be written uniquely in the form*

$$\epsilon = \zeta \epsilon_1^{v_1} ... \epsilon_r^{v_r}$$

with $\zeta \in \mu(K)$, $v_i \in \mathbb{Z}$, and $r = r_1 + r_2 - 1$ ($\mathbb{Z}_K^\times \cong \mu(K) \times \mathbb{Z}^r$).

Proof. Consider the map: $\lambda : K^\times \to \mathbb{R}^{r_1 + r_2}$ restrict to $\lambda : \mathbb{Z}_K^\times \to H$. Then the kernel is $\mu(K)$, image is Γ, and Γ is complete lattice in r-dimensional vector space. Therefore, $\Gamma \cong \mathbb{Z}^r$. $\square$

Definition 4.26. We define $\epsilon_1, ..., \epsilon_r$ as the*fundamental units.*

5. Calculate Class Number through Analysis

5.1. Riemann Zeta Function.

Definition 5.1. We define the *Riemann Zeta Function* as following:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Definition 5.2. We define the *Gamma function* as following:

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$$

Definition 5.3. We define the *functional equation* as following:

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

Definition 5.4. We define the *Dedekind zeta function* as following:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}$$

where $\mathfrak{a}$ is an integral ideal in number field K.

Fact 5.5. *We can write the Riemann Zeta Function as*

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)$$

Fact 5.6. *$\zeta(s), s \in \mathbb{R}$, converges absolutely for all $s > 1$, and diverges for $s \leq 1$.*

Fact 5.7. *If $Re(s) > 1$, then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}$$

The same applies to Dedekind zeta function: If $Re(s) > 1$, then

$$\zeta(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}}$$

In this section, we want to know a bit about the functional equation. Thus, we will begin proving a result:

Lemma 5.8. *Set $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi t^2 n^2}$. Then if $t \neq 0$, $\theta(1/t) = t\theta(t)$.*

Proof. Fix $t > 0$, and let $f(x) = e^{-\pi t^2 x^2}$. And define $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Thus $F(x) = \sum_{n\in\mathbb{Z}} e^{-\pi t^2 (x+n)^2}$. We know that $F(0) = \theta(t)$. It is periodic with $F(x) = F(x+1)$. We take its Fourier series. $F(x) = \sum_{m\in\mathbb{Z}} a_m e^{2\pi i m x}$. Thus, we compute a_m:

$$\begin{aligned} a_m &= \int_0^1 F(x) e^{-2\pi i m x} dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m (x+n)} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i m x} dx = \int_{-\infty}^{\infty} e^{-\pi t^2 x^2 - 2\pi i m x} dx \\ &= e^{-\pi m^2/t^2} \int_{-\infty}^{\infty} e^{-\pi (tx + im/t)^2} dx = t^{-1} e^{-\pi m^2/t^2} \end{aligned}$$

Therefore,

$$\theta(t) = F(0) = \sum_{m\in\mathbb{Z}} a_m = \sum_{m\in\mathbb{Z}} t^{-1} e^{-\pi m^2/t^2} = t^{-1}\theta(1/t)$$

□

Proposition 5.9. *For $Re(s) > 1$, we have*

$$\int_0^{\infty} (\theta(t) - 1) t^{s-1} dt = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

Proof. We evaluate the integral by changing variable $u = nt$ and $v = \pi u^2$

$$\begin{aligned} &2\int_0^{\infty} \sum_{n\geq 1} e^{-\pi t^2 n^2} t^{s-1} dx = 2\sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} e^{-\pi u^2} u^{s-1} du \\ &= 2\zeta(s) \int_0^{\infty} e^{-v} (v/\pi)^{s/2-1} (2\pi)^{-1} dv = \pi^{-s/2} \Gamma(s/2) \zeta(s) \end{aligned}$$

□

Theorem 5.10. *For $Re(s) > 1$, $\xi(s) = \xi(1-s)$*

Proof. We evaluate $\xi(s)$ by changing variable $u = 1/t$, and by Lemma 4.8 and proposition 4.9, we have

$$\begin{aligned}\xi(s) &= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_0^1 (\theta(t)-1)t^{s-1}dt \\ &= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty (u\theta(u)-1)u^{-s-1}du \\ &= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty u^{-s}(\theta(u)-1) + u^{-s} - u^{-s-1}du \\ &= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty u^{-s}(\theta(u)-1)du - \frac{1}{s} - \frac{1}{1-s}\end{aligned}$$

Therefore, we get

$$\int_1^\infty (\theta(t)-1)(t^{s-1}+t^{-s})dt - \frac{1}{s} - \frac{1}{1-s}$$

Which clearly satisfies $\xi(s) = \xi(1-s)$. □

5.2. **Class Number Formula.**

Remark 5.11. In this section, we are going to derive the *Analytic Class Number Formula*:

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}R_K h_K}{m|D_K|^{1/2}}$$

Where R_K is the regulator of K, h_K is the class number of K, and m is the number of roots of unity in K

Definition 5.12. Let $\epsilon_1, \ldots, \epsilon_r$ be a set of fundamental units, $r = r_1+r_2-1$. $\lambda : K \to \mathbb{R}^{r_1+r_2}$ be the logarithm mapping. The *regulator*, R_K is the absolute value of the determinant of any $r \times r$ minor in the $(r+1) \times r$-matrix with entries $\lambda_i(\epsilon_j)$.

Definition 5.13. A *cone* in $\mathbb{R}^n$ is a subset $X \subset \mathbb{R}^n$ such that if $x \in X$ and $\lambda \in \mathbb{R}_{>0}$, then $\lambda x \in X$

Proposition 5.14. *Let X be a cone in $\mathbb{R}^n$, $F : X \to \mathbb{R}_{>0}$, be a function satisfies: $F(\xi x) = \xi^n F(x)$, with $x \in X, \xi \in \mathbb{R}_{>0}$. Let $T = \{x \in X | F(x) \leq 1\}$ be bounded, with non-zero volume $v = vol(T)$. Let Γ be a lattice in $\mathbb{R}^n$, with $\Delta = vol(\Gamma)$. Then $Z(s) = \sum_{\Gamma\cap X} \frac{1}{F(x)^s}$ converges for $Re(s) > 1$ and* $\lim_{s\to 1}(s-1)Z(s) = \frac{v}{\Delta}$

Proof. First, we notice $vol(\frac{1}{r}\Gamma) = \frac{\Delta}{r^n}$. Let $N(r)$ be the number of points in $\frac{1}{r}\Gamma \cap T$, then $N(r)$ is also the number of points in $\{x \in \Gamma \cap X | F(x) \leq r^n\}$.

Then $v = vol(T) = \lim_{r\to\infty} N(r)\dfrac{\Delta}{r^n}$. Now we arrange $0 < F(x_1) \le F(x_2) \le ...$, and let $r_k = F(x_k)^{1/n}$. Then $N(r_k - \epsilon) < k \le N(r_k)$. Therefore, $\dfrac{k}{r_k^n} \le \frac{N(r_k)}{r_k^n}$. This gives $\lim_{r_k\to\infty} \dfrac{k}{r_k^n} = \lim_{k\to\infty} \dfrac{k}{F(x_k)} = \dfrac{v}{\Delta}$. Thus, $\forall \epsilon > 0$, $\exists k_0$, such that $\forall k \ge k_0$, we have $(\dfrac{v}{\Delta} - \epsilon)\dfrac{1}{k} < \dfrac{1}{F(x_k)} < (\dfrac{v}{\Delta} + \epsilon)\dfrac{1}{k}$. Thus,

$$(\frac{v}{\Delta} - \epsilon)^s \sum_{k=k_0}^{\infty} \frac{1}{k^s} < \sum_{k=k_0}^{\infty} \frac{1}{F(x_k)^s} < (\frac{v}{\Delta} + \epsilon)^s \sum_{k=k_0}^{\infty} \frac{1}{k^s}$$

Therefore, we know it converges when $Re(s) > 1$. And if we multiply $\lim_{s\to 1}(s-1)$ on both sides, since the pole of $\zeta(s)$ is at $s = 1$. We will get the equation we want. □

Definition 5.15. The cone $X \subset K_{\mathbb{R}}$ is defined with the following property ($x \in X$):
(i) $N(x) \neq 0$
(ii) The coefficients ξ_i of $l(x)$ satisfy $0 \le \xi_i < 1$
(iii) $0 \le arg(x_1) < \dfrac{2\pi}{m}$, where x_1 is the first component of x

Lemma 5.16. *If $y \in \mathbb{R}^n$, with $N(y) \neq 0$. Then y is uniquely of the form $xi(\epsilon)$, where $x \in X$ and $\epsilon \in \mathbb{Z}_K^{\times}$.*

Proof. Let $l(y) = \gamma\lambda + \gamma_1\lambda(\epsilon_1) + ... + \gamma_r\lambda(\epsilon_r)$. Let's write $\gamma_i = k_i + \xi_i$, with $k \in \mathbb{Z}, \xi \in [0, 1)$. And let $\eta = \epsilon_1^{k_1}...\epsilon_r^{k_r}$. Let $z = yi(\eta)$. We know that $0 \le arg(z_1) - \dfrac{2k\pi}{m} < \dfrac{2\pi}{m}$ for some k. Choose $\zeta \in \mu(K)$ such that $\tau_1(\zeta) = e^{2\pi i/m}$, then $x = yi(\eta^{-1}\zeta^{-k}) \in X$, thus $y = xi(\eta\zeta)$. □

Remark 5.17. Let connect what we have proved before with class numbers. By Dedekind zeta function: $\zeta_K = \sum_{C\in C_K} f_C(s)$, summing all the ideal classes. And $f_C(s) = \sum_{\mathfrak{a}\in C} \dfrac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}$. But if we take $\mathfrak{b} \in C^{-1}$, then $\mathfrak{ab}$ is principal, say $\langle\alpha\rangle$. Thus, $\mathfrak{a}$ and $\langle\alpha\rangle$ are bijective, and $\alpha \in \mathfrak{b}$. Thus, $f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b})^s \sum_{\mathfrak{b}|\langle\alpha\rangle} \dfrac{1}{N_{K/\mathbb{Q}}(\alpha)^s}$. Let $\Gamma = i(\mathfrak{b})$, $\Theta = \{x \in K_{\mathbb{R}} | x = i(\alpha), \alpha \in \mathfrak{B}\}$, where $\mathfrak{B}$ is a complete set of non-associate members of $\mathfrak{b}$, then $f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b})^s \sum_{x\in\Theta} \dfrac{1}{N(x)^s}$

Proposition 5.18.

$$vol(T) = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{m}$$

Proof. Let $\epsilon \in \mathbb{Z}_K^{\times}$. Then it preserves volume. And also by the last lemma, we let $\tilde{T} = \bigcup_{k=0}^{m-1} Ti(\zeta^k)$, and this has $vol(\tilde{T}) = m \cdot vol(T)$. Now, we let

$\overline{T} = \{x \in \tilde{T} | x_i > 0, \forall i = 1, ..., r_1\}$. Then $vol(T) = \frac{2^{r_1}}{m} vol(\overline{T})$. Now we compute $vol(\overline{T})$ by change of variables first.

$$(x_1, ..., x_{r_1}, z_1, ..., z_{r_2}) \mapsto (x_1, ..., x_{r_1}, R_1, \phi_1, ..., R_{r_2}, \phi_{r_2})$$

Where $z_k = R_k e^{i\phi_k}$. And the Jacobian of this change is $R_1...R_{r_2}$. Now, since $l(x) = \xi\lambda + \xi_1\lambda(\epsilon_1) + ... + \xi_r\lambda(\epsilon_r)$, and

$$l(x_1, ..., x_{r_1}, z_1, ..., z_{r_2}) \mapsto (\log(x_1), ..., \log(x_{r_1}), 2\log(R_1), ..., 2\log(R_{r_2}))$$

. We could do another change of variable with $\log(x_i) = \frac{1}{n} \log\xi + \sum_{k=1}^{r} \xi_k \lambda_i(\epsilon_k)$ and $\log(R_i) = \frac{2}{n} \log\xi + \sum_{k=1}^{r} \xi_k \lambda_{r_1+i}(\epsilon_k)$. And the Jacobian is computed to be $|J| = \frac{R_K}{2^{r_2} R_1 ... R_{r_2}}$. Therefore,

$$\begin{aligned} vol(\overline{T}) = 2^{r_2} vol_{\mathbb{R}}(\overline{T}) &= 2^{r_2} \int_{\overline{T}} dx_1...dx_{r_1} dy_{r_1+1} dz_{r_1+1}...dy_{r_1+r_2} dz_{r_1+r_2} \\ &= 2^{r_2} \int_{\overline{T}} R_1...R_{r_2} dx_1...dx_{r_1} dR_1 d\phi_1 ... dR_{r_2} d\phi_{r_2} \\ &= 2^{r_2} (2\pi)^{r_2} \int_{\overline{T}} |J| R_1...R_{r_2} d\xi_1...d\xi_r = 2^{r_2} \pi^{r_2} R_K \end{aligned}$$

Thus, we plug this back into our equation. We will then get what we want. □

Let's make a conclusion with our remark.

Remark 5.19.

$$\lim_{s\to 1}(s-1) f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b}) \frac{v}{\Delta} = \frac{N_{K/\mathbb{Q}}(\mathfrak{b}) 2^{r_1+r_2} \pi^{r_2} R_K}{N_{K/\mathbb{Q}}(\mathfrak{b}) m |D|^{1/2}} = \frac{2^{r_1+r_2} \pi^{r_2} R_K}{m |D|^{1/2}}$$

Therefore, we could use the relation between $\zeta(s)$ and $f_C(s)$ to get:

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1+r_2} \pi^{r_2} R_K h_K}{m |D_K|^{1/2}}$$

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