CLASS NUMBER THROUGH ADÈLES

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Contents

1. Introduction

In this paper, we are going to prove the Dirichlet's Unit Theorem and the finiteness of ideal class group through adèles and idèles. This introduces a more topological method to prove the results in algebraic number theory by using knowledge in local field theory and global field theory.

2. $\mathfrak{p}$-adic Numbers

2.1. Norms and $\mathfrak{p}$-adic Integers.

Definition 2.1. Let F be a field, $x, y \in F$, a map $\|\cdot\| : F \to \mathbb{R}_{\geq 0}$ satisfying the following properties is called a *norm*:

(i) $\|x\| = 0$ if and only if $x = 0$
(ii) $\|x \cdot y\| = \|x\| \cdot \|y\|$
(iii) $\|x + y\| \leq \|x\| + \|y\|$

Definition 2.2. A norm is called *non-Archimedean* if $\|x+y\| \leq \max(\|x\|, \|y\|)$

Definition 2.3. Let F be a field, $x, y \in F$, a *valuation* is a map $v : F \to \mathbb{R}$ with the following properties:

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(i) $v(x) = \infty$ if and only if $x = 0$
(ii) $v(xy) = v(x) + v(y)$
(iii) $v(x + y) \geq \min(v(x), v(y))$

Definition 2.4. A valuation v is called *discrete* if the image of v is $\alpha\mathbb{Z}$ for some $\alpha \geq 0$

Definition 2.5. If k is a number field, O_k its ring of integers, $\mathfrak{p}$ a prime ideal of O_k, We call the map $v_\mathfrak{p} : k^\times \to \mathbb{Z}$ the $\mathfrak{p}$-*adic evaluation* if p generates $\mathfrak{p}$, $a \in k$, $a = up^m$, where u is a unit. Then $v_\mathfrak{p}(a) = m$ or $v_\mathfrak{p}(a) = \infty$ when $a = 0$.

Definition 2.6. Under the same construction as Def. 2.5, let f be the degree of the residue field extension $O_k/\mathfrak{p}$ over $\mathbb{Z}/(p)$, $\|\cdot\|_{v_\mathfrak{p}} : k \to \mathbb{R}$ is called the $\mathfrak{p}$-adic norm, if $\|a\|_{v_\mathfrak{p}} = \|\mathfrak{p}\|^{-fv_\mathfrak{p}(a)}$, or $\|a\|_{v_\mathfrak{p}} = 0$ when $a = 0$

We note that $\|\cdot\|_{v_\mathfrak{p}}$ is a non-archimedean norm.

Definition 2.7. We define $O_\mathfrak{p}$ to be the localization of O_k at $\mathfrak{p}$. Thus $O_\mathfrak{p} = \{\frac{a}{b} | b \notin \mathfrak{p}, a, b \in O_k\}$

Consider $M_n = O_\mathfrak{p}/(p^n)$, and the map $\varphi_m : M_n \to M_{n-1}$, by $\varphi(m_n) = m_n \mod p^{n-1}$, and we see that $M_{n-1} \hookrightarrow M_n$ and that $\{M_n\}$ forms an projective system.

Definition 2.8. We define the inverse limit $\varprojlim M_n$ to be the $\mathfrak{p}$-adic integers, we denote as O_v.

Remark 2.9. $O_k \subseteq O_\mathfrak{p} \subseteq O_v$. This is easy to check, $O_k \subseteq O_\mathfrak{p}$ is clear since one can pick $b = 1$. Consider $a, b \in O_k$, then $a, b \in O_v$ by map $a \mapsto M_{v_\mathfrak{p}(a)}$ and similarly for b. Then one can do long division to see that $a/b \in O_v$. Thus $O_\mathfrak{p} \subseteq O_v$

Definition 2.10. We define A_k to be the *canonical set* of k, which is a set of all the absolute values, one non-archimedean absolute value for each $\mathfrak{p}$, and an archimedean absolute value for each real and complex embedding. We denote the set of archimedean absolute values as S_∞.

2.2. **Completion of O_v.** In this section and throughout the rest of the paper, we denote $v = v_\mathfrak{p}$.

Remark 2.11. We observe that the units of O_v are the elements of O_v with valuation 0, otherwise they all divide p, the uniformizing element. Therefore, we get the field of fraction of O_v by inverting p. We write it as $K_v = O_v(p^{-1})$

Definition 2.12. The *basic open sets of K_v* are defined as the open balls $B(x, r) = \{y \in K_v | \|x - y\|_v < r\}$

Proposition 2.13. *The basic open sets of K_v are closed, and K_v is totally disconnected.*

Proof. It suffices to check that $B(0,r)$ are closed. Consider the image of $\|a\|_v = \|\mathfrak{p}\|^{-v_\mathfrak{p}(a)}$, where $v_\mathfrak{p}$ has integer image. Thus $im(\|\cdot\|_v) = \{\|\mathfrak{p}\|^{\mathbb{Z}}\}$. Therefore, we see that the image is discrete besides at 0, which is a limit point. Thus, for some small $\varepsilon > 0$, $B(0,r) = \overline{B(0,r-\varepsilon)}$. Therefore, the basic open sets are closed. And we consider $\cap_{r>0} B(x,r) = \{x\}$. Therefore, K_v is totally disconnected. □

Theorem 2.14. *K_v is a locally compact, complete topological field with compact open and closed subsets O_v and O_v^**

Proof. It is clear that K_v is a topological field, considering term by term addition and multiplication to be continuous. Completeness is easy to check since one can check if convergent sequences are Cauchy. Since K_v is discrete, and every point is contained in a ball, which is finite. Thus, each basic open sets is compact, K_v is locally compact.

O_v is finite since each M_n is discrete and finite, thus compact. By Tychonoff theorem, $O_v = \prod_{n=1}^{\infty} M_n$ is compact. O_v^* is compact since it is a closed subset of O_v, which is because $O_v^* = O_v - \mathfrak{p}_v = O_v - B(0,1)$. Then we prove O_v is closed. Consider it as the subspace of $\prod_{i=1}^{\infty} M_i$, if $(a_m) \notin O_v$, then $\exists N > 0$ such that $\varphi_N(a_N) \neq a_{N-1}$. Thus $(a_m) \in (a_1, \ldots, a_{N-1}) \times \prod_{m>N} M_m$, which is open. Thus O_v is the complement of an open set, thus closed. □

Proposition 2.15. *O_v^*, O_v, K_v are topological closures of O_k^*, O_k, k respectively.*

3. Adeles and Ideles

3.1. Properties of Adeles.

Definition 3.1. We define *adeles* $\mathbb{A}_k$ to be the subset of $\prod_{v \in A_k} k_v$, such that $\mathbb{A}_k = \{(a_v)_{v \in A_k} | a_v \in O_v \text{ for all but finitely many } v\}$

Remark 3.2. $\mathbb{A}_K$ inherits the subspace topology from $\prod_{v \in A_k} K_v$, to be specific, for some $S_\infty \subset S \subset A_k$, the basic open sets of $\mathbb{A}_K$ is the sets $\prod_{v \in S} U_v \times \prod_{v \notin S} O_v$, where U_v is open in K_v.

Proposition 3.3. *$\mathbb{A}_K$ is a locally compact topological ring.*

Proof. We first show $\mathbb{A}_K$ is locally compact. Since the basic open sets of $\mathbb{A}_K$ are in the form $\prod_{v \in S} U_v \times \prod_{v \notin S} O_v$, we know that $\mathbb{A}_K \subset \prod_{v \in S} K_v \times \prod_{v \notin S} O_v$. Since each O_v is compact, $\prod_{v \notin S} O_v$ is compact by Tychonoff theorem, and each K_v is locally compact, thus $\prod_{v \in S} K_v$ is locally compact. Therefore, we know that $\prod_{v \in S} K_v \times \prod_{v \notin S} O_v$ is locally compact, thus $\mathbb{A}_K$ is locally compact.

Next, $\mathbb{A}_K$ is a ring is clear. We check continuity. Consider $f(a,b) = a + b$. Pick a basic open set $U = \prod_{v \in S} B(c_v, r_v) \times \prod_{v \notin S} O_v$. If $(a_v, b_v) \in f^{-1}(U)$. Then $\|a_v + b_v - c_v\|_v < r_v$. Next pick the sets $U_1, U_2 \subset A$ such that $a_v \in U_1, b_v \in U_2, f(U_1), f(U_2) \subset U$, then $U_1 = \prod_{v\in S} B(a_v, r_v/4) \times \prod_{v\notin S} O_v$ and $U_2 = \prod_{v\in S} B(b_v, r_v/4) \times \prod_{v\notin S} O_v$ works, since $f(U_1 \times U_2) = \prod_{v\in S} B(a_v + b_v, r_v/2) \times \prod_{v\notin S} O_v \subset \prod_{v\in S} B(c_v, r_v) \times \prod_{v\notin S} O_v$. Similarly, consider $f(a, b) = ab$, and the balls with radius $\sqrt{r_v/2}$. □

Proposition 3.4. *Let $\iota : k \hookrightarrow \mathbb{A}_K$ such that $\iota(x) = x_v$, then $\iota(k)$ is a discrete subring of $\mathbb{A}_K$*

Proof. k is a subring is clear. We prove that it is discrete. Consider the basic open set at 0: $U = \prod_{v\in S_\infty} B(0, 1/2) \times \prod_{v\notin S_\infty} O_v$. Assume $0 \neq \alpha \in U \cap k$, then $\|\alpha\|_v \leq 1$ for non-achimedean evaluation and $\|\alpha\|_v < 1/2$ for archimedean ones. Thus $\prod_{v\in A_k} \|\alpha\|_v^{n_v} < 1/2$. But $\prod_{v\in A_k} \|\alpha\|_v^{n_v} = 1$ by product formula. Thus contradiction. Thus $U \cap k = 0$. Thus $\iota(k)$ is discrete. □

Proposition 3.5. *Consider $\lambda : K_v \hookrightarrow \mathbb{A}_K$, if $x \in K_v$, then $\lambda(x) = (0, \ldots, 0, x, 0, \ldots) \in \mathbb{A}_K$. $\lambda(K_v)$ a closed subring, and it inherits the usual topology on K_v.*

Proof. The inheritance of the topology is clear. Now consider that $\lambda(K_v)$ is closed. Assume $v' \neq v, a_{v'} \neq 0$. Consider the following open set of $a \in \mathbb{A}_K - K_v$: $B(a_v, \|a_v\|/2) \times \prod_{A_k - S_\infty, n\neq v'} O_n \times \prod_{n\in S_\infty} K_n$. This is disjoint to $\lambda(K_v)$. Thus $\lambda(K_v)$ is closed. □

Next, we are going to show that $\mathbb{A}_K/k$ is compact.

Lemma 3.6. *$k + \mathbb{A}_K^{S_\infty} = \mathbb{A}_K$, where $\mathbb{A}_K^{S_\infty}$ is the set $\prod_{v\in S_\infty} K_v \times \prod_{v\notin S_\infty} O_v$. Thus we have any adele can be written as a sum of an element of k and $\mathbb{A}_K^{S_\infty}$.*

Proof. We prove if $a \in \mathbb{A}_K$, then $\exists \alpha \in k$ such that $\forall v \notin S_\infty, a - \alpha \in O_v$. Since $a_v \in O_v$ for all but finitely many $v \notin S_\infty$, so we take c such that c is highly divisible by finitely many primes of $\mathbb{Z}$ lying under $\mathfrak{p}$ with $a_v \notin O_{v_\mathfrak{p}}$. Therefore, $\exists c$, such that $ca \in O_v$. Then let S be the set of primes of O_k dividing cO_k, by approximation theorem, let $\alpha \in O_k$ be $\alpha \equiv ca_v \mod \mathfrak{p}^m$, for $\mathfrak{p} \in S$ and large m. Therefore, if $\mathfrak{p} \notin S$, then $c \in O_v^*$, so $\alpha/c \in O_v$, $a_v - \alpha/c \in O_v$. If $\mathfrak{p} \in S$, then if m is large enough, $a_v - \alpha/c \in O_v$. □

Theorem 3.7. *$\mathbb{A}_K/k$ is compact.*

Proof. Consider the restricted mapping $\iota|_{K_v} : k \hookrightarrow \prod_{v\in S_\infty} K_v = \mathbb{R}^n$. And O_v, as the $\mathfrak{p}$-adic integers, forms the lattice. Let $P \in \mathbb{R}^n$ be a fundamental parallelotope for the lattice O_k, since it has the same rank as $\mathbb{R}^n$, P is bounded and $\bar{P}$ is compact. Also, we proved that O_v is compact. By Tychonoff theorem, $Q = \prod_{v\notin S_\infty} O_v \times \bar{P}$ is compact. By lemma 3.6, $a - \alpha \in \mathbb{A}_K^{S_\infty}$, thus we translate by β to get $a - \alpha + \beta \in \prod_{v\notin S_\infty} O_v \times \bar{P}$, and $-\alpha + \beta \in k$ which translates a into a compact set. Thus $A/k = kA/k = kQ/k \cong Q/(k \cap Q)$ which is compact. □

3.2. **Properties of Ideles.**

Definition 3.8. We define *ideles* $\mathbb{I}_K$ to be the subset of $\prod_{v\in A_k} K_v^*$, such that $\mathbb{I}_K = \{(a_v)_{v\in A_k} | a_v \in O_v^* \text{ for all but finitely many } v\}$.

Remark 3.9. One can see that the ideles is the units in adeles, and if we define $\|\cdot\|_{\mathbb{A}_K} : \mathbb{A}_K^* \to \mathbb{R}_{>0}$, with $\|a\|_{\mathbb{A}_K} = \prod_{v\in A_v} \|a_v\|_v$. Then the ideles is the kernel of this map. However, since the adeles forms a ring, the units of a ring doesn't need to be a multiplication group since the inverse doesn't need to be continuous.

Remark 3.10. One has two ways to consider the basic open set: $\prod_{v\in S} U_v \times \prod_{v\notin S} O_v^*$, where $U_v \subset K_v^*$, which is inherited from adelic ring. One can also consider $\phi : \mathbb{I}_K \to A \times A$, where $\phi(a) = (a, a^{-1})$. We observe that ϕ is a homeomorphism. Let $U = (\prod_{v\in S} U_v \times \prod_{v\notin S} O_v^*, \prod_{v\in S} U_v \times \prod_{v\notin S} O_v^*)$. Then $U \cap \phi(\mathbb{I}_K) = (\prod_{v\in S} U_v \cap (U_v')^{-1} \times \prod_{v\notin S} O_v^*, \prod_{v\in S}(U_v^{-1} \cap U_v') \times \prod_{v\notin S} O_v)$, whose preimage is $\prod_{v\in S} U_v \cap (U_v')^{-1} \times \prod_{v\notin S} O_v^*$, which is open. Thus we have ϕ is homeomorphic.

Proposition 3.11. *$\mathbb{I}_K$ is a topological group.*

Proof. Since $\mathbb{I}_K$ is the kernel of our normed map, it is clearly a group.

Consider $\phi : \mathbb{I}_K \hookrightarrow A \times A$. Since we have shown ϕ is a homeomorphism, ϕ^{-1} is continuous, thus the inversion map is continuous.

Then define $\psi : \mathbb{I}_K \times \mathbb{I}_K \to \mathbb{I}_K$ by pointwise multiplication seeing $\mathbb{I}_K$ as in $A \times A$. Then let $(a_1, a_2), (a_3, a_4) \in \mathbb{I}_K \subset A \times A$, then $(a_1, a_2)(a_3, a_4) = (a_1a_3, a_2a_4)$ is continuous since A is an adelic ring, thus a topological ring. □

Proposition 3.12. *Let $\iota : k^* \hookrightarrow \mathbb{I}_K$ such that $\iota(x) = x_v$, then $\iota(k^*)$ is a discrete subgroup of $\mathbb{I}_K$*

Proof. Consider the basic open set at 1: $U = \prod_{v\in S_\infty} B(1, 1/2) \times \prod_{v\notin S_\infty} O_v^*$. Assume $0 \neq \alpha \in U \cap k^*$, then $\|\alpha - 1\|_v \leq 1$ for non-archimedean evaluation and $\|\alpha - 1\|_v < 1/2$ for archimedean ones. Thus $\prod_{v\in A_k} \|\alpha - 1\|_v^{n_v} < 1/2$, which contradicts the product formula. Thus $U \cap k^* = 1$. Thus $\iota(k)$ is discrete. □

Proposition 3.13. *Condier $\lambda : K_v^* \hookrightarrow \mathbb{I}_K$, if $x \in K_v^*$, then $\lambda(x) = (0, \ldots, 0, x, 0, \ldots) \in \mathbb{I}_K$. $\lambda(K_v^*)$ a closed subgroup.*

Proof. Let $v' \neq v$, $a_{v'} \neq 0$, then if $a \in \mathbb{I}_K - K_v^*$, then consider $B(a_v, \|a_v\|/2)| \times \prod_{A_k - S_\infty, n\neq v'} O_n^* \times \prod_{n\in S_\infty} K_n^*$, which is an open neighborhood disjoint from $\lambda(K_v^*)$. Thus $\lambda(K_v^*)$ is closed. □

4. Applications in Algebraic Number Theory

4.1. **The Idele Class Group.**

Remark 4.1. We define a map: $\|\cdot\| : \mathbb{I}_K \to \mathbb{R}^+$ with $\|a\| = \prod_{v\in A_k} \|a_v\|_v^{n_v}$

Lemma 4.2. $\|\cdot\|$ *is continuous.*

Proof. Assume $(b_0, b_1) \subset \mathbb{R}^+$. We show that the preimage is open in $\mathbb{I}_K$. If $a \in \mathbb{I}_K$, and $\|a\| \in (b_0, b_1)$, then pick an archimedean valuation v_0. Let S be the set of valuations where a is not in O_v^*, consider the open sets $U_r = \prod_{v \in S, v \neq v_0} B(a_v, 1) \times \prod_{v \notin S} O_v^* \times B(a_{v_0}, r)$ where r varies. Thus $a \in U_r$. Then we can make r small so that $\|U_r\|$ lies in (b_0, b_1). □

Remark 4.3. Since $\{1\}$ is closed, $ker(\|\cdot\|)$ is closed in $\mathbb{I}_K$. We call this $\mathbb{I}_K^1$. By product formula, $k^* \subseteq \mathbb{I}_K^1$, is a discrete subgroup.

Now, let $\mathfrak{F}_K$ be the multiplication group of fractional ideals of k. Let $\mathfrak{P}\mathfrak{F}_K$ be the subgroup of principal ideals. Let $C_K = \mathfrak{F}_K / \mathfrak{P}\mathfrak{F}_K$ be the class group over k.

Theorem 4.4. $\mathbb{I}_K / k^* \mathbb{I}_K^{S\infty} \cong C_K$

Proof. We define $\phi : \mathbb{I}_K \to \mathfrak{F}_K$ by if $\phi(a) = \prod_{v \in A_k - S_\infty} \mathfrak{p}^{v_\mathfrak{p}(a_v)}$. Thus, only finitely many $v_\mathfrak{p}(a_v)$ are non-zero, indicating that this is a fractional ideal. This map is surjective, since one can find the preimage easily for each element. The kernel of this map happens when all $v_\mathfrak{p}(a_v) = 0$, but the archimedean entries can vary. Thus the kernel is $\mathbb{I}_K^{S_\infty}$. Moreover, if $\alpha \in k^*$, then $\phi(\alpha)$ is a principal ideal generated by (α). Thus $\phi(k^*) \subset \mathfrak{P}\mathfrak{F}_K$. Thus $\phi(k^*) = \mathfrak{P}\mathfrak{F}_K$. Thus, there is an induced surjective homomorphism from $\psi : \mathbb{I}_K / k^* \to C_K$. And consider $\varphi : \mathbb{I}_K \to C_K$, then $ker(\varphi) = k^* \mathbb{I}_K^{S_\infty}$ by the discussion above. Thus by the first isomorphism theorem, $\mathbb{I}_K / k^* \mathbb{I}_K^{S\infty} \cong C_K$ □

Definition 4.5. We call $C = \mathbb{I}_K / k^*$ the idele class group, and $C^1 = \mathbb{I}_K^1 / k^*$.

Definition 4.6. Let S be a finite subset of A_k containing S_∞, we call $k_S = \mathbb{I}_K^S \cap k^*$ the *S-units* of k. And we call $\mathbb{I}_K^S / k_S$ the group of *S-idele classes* and denote by C_S. We set $\mathbb{I}_K^{S1} = \mathbb{I}_K^S \cap \mathbb{I}_K^1$ and $C_S^1 = \mathbb{I}_K^{S1} / k_S$

Remark 4.7. For each S, $C_S \hookrightarrow C$. Since $\mathbb{I}_K^S$ is both open and closed in $\mathbb{I}_K$, C_S is both open and closed in C. Similarly, $C_S^1 \hookrightarrow C$, and C_S^1 is both open and clased in C^1. Then consider the map $C \to C_K$, then the kernel will be C_{S_∞} since $\mathbb{I}_K / k^* \mathbb{I}_K^{S\infty} \cong C_K$, by third isomorphism theorem, $\mathbb{I}_K / k^* \mathbb{I}_K^{S\infty} \cong \frac{\mathbb{I}_K}{k^*} / \frac{\mathbb{I}_K^{S\infty} k^*}{k^*}$, and the latter is isomorphic to C_{S_∞} by the second isomorphism theorem. Thus $C / C_{S\infty} \cong C_K$

Finally, notice the map $C^1 \to C_K$ is also surjective, since if $\mathfrak{f} \in \mathfrak{F}$, $a \in \mathbb{I}_K$ with $\phi(a) = I$, we can change an archimedean value to get $a' \in \mathbb{I}_K^1$, and $\phi(a') = I$. Thus the kernel is $C_{S_\infty}^1$. So we have $C^1 / C_{S\infty}^1 \cong C_K$

4.2. Approximation Theorem.

Definition 4.8. Let $a \in \mathbb{A}_K$, we define $L(a) \subseteq k$, such that $L(a) = \{\alpha | \|\alpha\|_v \leq \|\alpha_v\|_v, \alpha \in k, \forall v \in A_k\}$. We write $\lambda = |L(a)|$.

Remark 4.9. If $\alpha \in k^*$, $x \in L(a)$, then since $\|x\|_v \leq \|x_v\|_v$, then $\|\alpha x\|_v = \|\alpha\|_v \|x\|_v \leq \|\alpha\|_v \|x_v\|_v = \|\alpha x_v\|_v$. Therefore, there is a bijection between $L(a)$ and $L(\alpha a)$, thus $\lambda(a) = \lambda(\alpha a)$

Theorem 4.10. *$\exists c_0$ a constant, depending only on k such that for any $a \in \mathbb{A}_K$, $\lambda(a) \geq c_0 \|a\|$.*

Proof. For the proof of this theorem, refer to [1]Theorem 8.1. □

Lemma 4.11. *Let $a \in \mathbb{I}_K$, $\|a\| \geq 2/c_0$, then $\exists \alpha \in k^*$ such that we have $\forall v \in A_k, 1 \leq \|\alpha a_v\|_v \leq \|a\|$*

Proof. For the proof of this lemma, refer to [1]Lemma 9.1 □

4.3. **Finiteness of Class Number.**

Theorem 4.12. *C^1 is compact*

Proof. Let $\psi : \mathbb{I}_K \to \mathbb{R}^+$ by $\psi(a) = \|a\|$. Since if $a \in k^*$, $\phi(a) = 1$ by the product formula. Thus consider $\psi : C \to \mathbb{R}^+$. Thus $ker(\psi) = C^1$. Then since if $\rho \in \mathbb{R}^+$, $a_\rho \in \mathbb{I}_K$ with $\|a_\rho\| = \rho$, then $\psi^{-1}(\rho) = a_\rho C^1$. Therefore, we see that ψ^{-1} is homeomorphic to C^1. Thus we prove ψ^{-1} is compact.

Let $\rho > 2/c_0$, pick $a \in \psi^{-1}(\rho)$, then by Lemma 4.11, $\exists \alpha_a \in k^*$ such that $\forall v \in A_k, 1 \leq \|\alpha_a a_v\|_v \leq \rho$. Since $\|\cdot\|_v$ cannot take values bewteen 1 and $N(\mathfrak{p})$, and there are only finitely many $\mathfrak{p}$ with $N(\mathfrak{p}) \leq \rho$. Thus we have $\|\alpha_a a_v\|_{v_\mathfrak{p}} = 1$ for all but finitely many of $v_\mathfrak{p}$. Thus $\exists S \supset S_\infty$ such that $1 \leq \|\alpha_a a_v\|_v \leq \rho$ if $v \in S$ but $\|\alpha_a a_v\|_v = 1$ if $v \notin S$.

Now define $T = \prod_{v \in S}(\overline{B(0,\rho)} - B(0,1)) \times \prod_{v \notin S} O_v^*$. Thus by Tychonoff theorem, T is compact, since $\psi^{-1}(\rho)$ is a close subset of $\phi(T)$, where $\phi : \mathbb{I}_K \to C$. Thus ψ^{-1} is compact. □

Corollary 4.13. *C_S^1 is compact for any finite set S containing S_∞.*

Theorem 4.14 (Finiteness of class number)**.** *For any number field k, C_K is finite.*

Proof. Since $C^1/C^1_{S_\infty} \cong C_K$, and C^1 is compact, thus C_K is compact. Since $C^1_{S_\infty}$ is open, C_K is also discrete. Thus C_K is both compact and discrete, thus it is finite. □

Lemma 4.15. *Any discrete subgroup Λ of $\mathbb{R}^s$ is free abelian of rank* $\dim \mathbb{R}\Lambda$

Proof. We induct on $\dim \mathbb{R}\Lambda$. If $\dim \mathbb{R}\Lambda = 1$, since Λ is discrete, $\exists \lambda \in \Lambda$ closest to 0. Thus $\Lambda = \mathbb{Z}\lambda$.

Assume $\dim \Lambda = m$, let $\lambda_1, \dots, \lambda_m$ be a $\mathbb{R}-$basis for $\mathbb{R}\Lambda$. If Λ_0 is a subgroup of Λ spanned by $\lambda, \dots, \lambda_{m-1}$, then by induction hypothesis, $\Lambda_0 = \mathbb{Z}\lambda_1 \oplus \dots \oplus \mathbb{Z}\lambda_{m-1}$. Let Λ' of $\lambda \in \Lambda$ such that $\lambda = a_1\lambda_1 + \dots + a_m\lambda_m$ with $0 \leq a_i < 1$ for $i \leq m-1$ and $0 \leq a_m \leq 1$. Then Λ' is bounded, thus finite. Let $\lambda' \in \Lambda'$ has minimal nonzero coefficient, assume $\lambda' = a_1'\lambda_1 + \dots + a_m'\lambda_m$.

Assume if $\lambda \in \Lambda$, then $\exists t$ such that the mth coefficient of $\lambda - t\lambda'$ gives $0 \leq a_m < a'_m$. Then we modify by $\lambda_0 \in \Lambda_0$ to get $\lambda - t\lambda' - \lambda_0 \in \Lambda'$. Thus $a_m = 0$ since a'_m is minimal. Thus $\lambda - t\lambda' - \lambda_0 = 0$. Since $\lambda_1, \ldots, \lambda_m$ are linearly independent, we have $\Lambda = \mathbb{Z}\lambda \oplus \ldots \oplus \mathbb{Z}\lambda_m$. □

Theorem 4.16 (Dirichlet Unit Theorem)**.** *For any set finite set $S \in A_k$ of size s containing S_∞, the S-units k_S have rank $s-1$.*

Proof. Let $v_1, \ldots, v_s \in S$, and assume v_s is archimedean. Define $\log : \mathbb{I}_K^S \to \mathbb{R}^s$ by $\log(a) = (\log \|a_{v_1}\|_{v_1}^{n_{v_1}}, \ldots, \log \|a_{v_s}\|_{v_s}^{n_{v_s}})$. Since it is continuous in each coordinate, log is continuous. Also, since $a \in \mathbb{I}_K^{S1}$, then $\|a\| = 1$, since $\|a_v\|_v = 1$ for $v \notin S$, $\log(\mathbb{I}_K^{S1}) = \{x_1, \ldots, x_s \in \mathbb{R} | x_1 + \ldots + x_s = 0\} = H$. Thus $\dim \log(\mathbb{I}_K^{S1}) = s - 1$.

Since in a bounded region in $\mathbb{R}^s$, $\log(k_S)$ has bounded achidemean aboslute values. Thus the coefficients of the polynomials of these elements over $\mathbb{Z}$ is bounded. And the degree is bounded by $[k : \mathbb{Q}]$, there are finitely many such polynomials. Thus there are finitely many k maps into this bounded region. Thus $\log(k_S)$ is discrete. By the last lemma, $\log(k_S)$ is a free abelian group.

Let W be the subspace of H generated by $\log(k_S)$, then consider $\log : \mathbb{I}_K^{S1}/k_S = C_S^1 \to H/W$. Since $\mathbb{I}_K^{S1}$ generates H, the image generates H/W as an $\mathbb{R}$-vector space. Since log is continuous, C_S^1 is compact, the image is compact. Then if H/W is non-trivial, then it has no non-trivial compact subgroups. Thus $H/W = 0, H = W$, $\dim \log(k_S) = s - 1$. □

Corollary 4.17. *The group of global units of a number field k is isomorphic to $\mu(k) \times \mathbb{Z}^{r_1 + r_2 - 1}$, where $\mu(k)$ is the roots of unity, r_1 is the number of real embeddings, r_2 is the number of complex embeddings.*

References

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