Notes for Honors Analysis I

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Contents

Chapter 1

One Variable Calculus

1.1 Aug. 28, 2019

1.1.1 Uniform Structures - Interchanging Limits

Let $a_{ij} \in \mathbb{R}$ be a double sequence, in general, limits cannot be interchanged:

$$\lim_i \lim_j a_{ij} \neq \lim_j \lim_i a_{ij}$$

Example 1.1.1.

1. $a_{ij} = \frac{i}{i+j}, i, j \in \mathbb{N}$
2. $f(x) = \sum_{k=0}^{\infty} \frac{x^2}{(1+x^2)^k}$. We observe that f is not continuous at $x = 0$, and each $f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}$ is continuous, so that we have

   $$\lim_{x\to 0} \lim_{n\to\infty} f_n(x) = \lim_{x\to 0} f(x) \neq \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to 0} f_n(x)$$

   One can compute the geometric series to observe the continuous fact:

   $$f(x) = \begin{cases} 0, x = 0 \\ 1 + x^2, x \neq 0 \end{cases}$$

   Notice that f_n is continuous in $\mathbb{R}$ and $f(x) = \lim_{n\to\infty f_n(x)}$ exists for all $n \in \mathbb{R}$ but f need not be continuous.
3. For differential functions: if f_n is differentiable on $\mathbb{R}$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x, and f is differentiable on $\mathbb{R}$, yet $\lim_{n\to\infty} f'(x)$ need not equal $f'(x)$.

Definition 1.1.2. Let $f_n, f : S \to \mathbb{R}$, where S is a non-empty set. f_n *converges uniformly* to f in S if $\forall \varepsilon > 0, \exists N > 0$ such that $\forall n \geq N, \forall x \in S, |f_n(x) - f(x)| < \varepsilon$, or we can denote $\|f_n - f\|_{\sup} := \sup_{x\in S} |f_n(x) - f(x)| < \varepsilon$.

Proposition 1.1.3. *Let $b_{ij} \in \mathbb{R}$ be a double sequence, $i, j \in \mathbb{Z}^+$, satisfying the following:*

1. $\forall i$, $\lim_j b_{ij} = L_j$ *exists in* $\mathbb{R}$

2. $\forall j$, $\lim_i b_{ij} = \tilde{L_j}$ *in* $\mathbb{R}$

Then the following holds:

1. $\lim_i b_{ij} = \tilde{L_j}$ *exists in* $\mathbb{R}$ *uniformly* $\forall j$

2. $\lim_i L_i = \lim_j \tilde{L_j}$ *(interchange of limits)*

3. $\forall \epsilon > 0, \exists i_0 > 0, \exists j_0 > 0$ *such that* $\forall i \geq i_0, \forall j \geq j_0, |b_{ij} - L_i| < \varepsilon$ *(double limit)*

Proof. (i) Fix $\varepsilon > 0$, from condition 1, we have $\forall i, \forall j_0 \geq j, |b_{ij} - L_i| \leq \varepsilon$, then from condition 2, we have $\forall j, \forall i_0 \geq i, |b_{ij} - \tilde{L_j}| < \varepsilon$. Therefore, $\forall j, j' \geq j_0, |\tilde{L_j} - \tilde{L'_j}| \leq |\tilde{L_j} - L_i| + |L_i - \tilde{L'_j}|$. Since $|\tilde{L_j} - L_i| \leq |\tilde{L_j} - b_{ij}| + |b_{ij} - L_i| < 2\varepsilon$, and similarly $|L_i - \tilde{L'_j}| \leq |L_i - b_{ij}| + |b_{ij} - \tilde{L'_j}| < 2\varepsilon$, we have $|\tilde{L_j} - \tilde{L'_j}| \leq 4\varepsilon$. Therefore, $\{\tilde{L_j}\}$ is Cauchy in $\mathbb{R}$ annd so converges to $\lim_j \tilde{L_j} = L$ for some $L \in \mathbb{R}$.

(ii) Take $j = j_0$. For some $L \in \mathbb{R}$, we have $\forall i \geq i_0(j_0), |L_i - L| \leq |L_i - b_{ij_0}| + |b_{ij_0} - \tilde{L_{j_0}}| + |\tilde{L_{j_0}} - L| < 6\varepsilon$. Therefore, $\lim_i L_i = L = \lim_j \tilde{L_j}$.

(iii) From above, we have $\exists j_0 > 0, \exists i_1 > 0$ such that $\forall i \geq i_1, \forall j \geq j_0, |b_{ij} - L_i| \leq |b_{ij} - L_i| + |L_i - L| < 2\varepsilon$ uniformly of the $\tilde{L_j}$ limit. Also, $\exists j_1 > 0, \exists i_0 > 0$ such that $\forall i \geq i_0, \forall j \geq j_0, |b_{ij} - \tilde{L_j}| \leq |b_{ij} - \tilde{L_j}| + |\tilde{L_j} - L| < 2\varepsilon$ uniformly of the L_i limit. □

1.2 Aug. 30, 2019

Remark 1.2.1. The above proposition can be used to derive e.g. the Leibniz test of the alternating series, the Weiestrauss M-test (for function series) and test for convergence of double series/iterated sums, e.g. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be the extended real line, if $a_{ij} \geq 0$, then $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.

1.2.1 An application to real-valued functions

Recall the Heine definition of continuity:

Definition 1.2.2. f is continuous at $x \in [a, b]$ if $\forall y_i \to x$, $f(y_i) \to x$.

Theorem 1.2.3. *Assume* $f_j : [a, b] \to \mathbb{R}$*, where* $-\infty < a < b < \infty$*,* $j \in \mathbb{Z}^+$*, and* f_j *is continuous at* $x \in [a, b]$*. If* $f_j \to f : [a, b] \to \mathbb{R}$ *as* $j \to \infty$ *uniformly on* $[a, b]$*, then* f *is continuous at* $x \in [a, b]$

Proof. The theorem follows directly from Proposition 1.1.3 under the following: Let $y_i \to x$ when $i \to \infty$ and set $b_{ij} = f_j(y_i)$. Then if f_j are continuous at $x \in [a, b]$, then (ii) of proposition tells us that $\lim_{i\to\infty} b_i j = \tilde{L_j}$ and that $f_j \to_{\mathrm{J}\to\infty} f$ uniformly in $x \in [a, b]$ then (i) of proposition holds $\lim_j b_{ij} = L_j$. Therefore it follows □

Remark 1.2.4. In general a typical argument goes like this: $|f(y) - f(x)| \leq |f(y) - f_j(y)| + |f_j(y) - f_j(x)| + |f_j(x) - f(x)| < 3\varepsilon$

1.3 Sep. 2, 2019

Example 1.3.1. This is a counterexample, consider $f_1 = x$, $f_2 = x^2$, $f_3 = x^3$ etc. We see that each fucntion f_n is continuous, $\forall n, f_n(1) = 1$. $\forall x \in [0, 1), \forall n, f_n \to 0$. But $f_\infty(x)$ is not continuous.

Question 1.3.2. What happends if we interchange the roles of n and x?

To be specific, this question equals to the following : what arbitrary convergence of f_j and f_j continuous in x uniformly in j?

Definition 1.3.3. Let $\mathcal{F}$ be a family of functions defined on $[a, b]$, $-\infty < a < b < \infty$.

1. $\mathcal{F}$ is *equicontinuous* on $[a, b]$ if $\forall x \in [a, b]$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall y \in [a, b]$, $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$, $\forall f \in \mathcal{F}$.
2. $\mathcal{F}$ is *pointwise bounded* on $[a, b]$ if: $\forall x \in [a, b]$, $\exists M > 0$ such that $|f(x)| \leq M$, $\forall f \in \mathcal{F}$
3. $\mathcal{F}$ is *uniformly equicontinuous* if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in [a, b]$, $\forall y \in [a, b]$, $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$, $\forall f \in \mathcal{F}$.
4. $\mathcal{F}$ is *uniformly bounded* on $[a, b]$ if: $\exists M > 0$ such that $\forall x \in [a, b]$, $|f(x)| \leq M$, $\forall f \in \mathcal{F}$

Theorem 1.3.4. *(Ascoli) $\{f_n\}_{n=1,2...}$: $f_n : [a, b] \to \mathbb{R}$ are equicontinuous and pointwise boundeced on $[a, b]$ implies*

1. *$\{f_n\}$ is uniformly equicontinuous on $[a, b]$*
2. *$\{f_n\}$ is uniformly bounded on $[a, b]$*
3. *$\{f_n\}$ has a uniformly converging subsequence.*

Remark 1.3.5. Because of the last conclusion, Ascoli can be viewed as a "compactness result" later on.

Do not confuse uniform convergence of the sequence of functions f_n with uniform continuity of a single function f (Recall definition: $f : [a, b] \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in [a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$).

Recall that f is uniformly continuous on $[a, b]$ implies f is continuous on $[a, b]$.

If f is continuous on $[a, b]$ which is closed and bounded, then f is uniformly continuous on $[a, b]$. And f attains its extreme values in $[a, b]$.

Proof to 1.3.4: First, observe that each f_n is uniformly continuous on $[a, b]$:

$\forall n > 0, \forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

And also each f_n attains its min/max mvalues in $[a, b]$, i.e. $\forall n > 0, \exists M_n > 0$ such that $|f_n(x)| \leq M$, $\forall x \in [a, b]$.

Out first two tasks is to show that the above hold uniformly for each n.

Uniform Bounded: if not, then $\forall n$, $\exists x_n \in [a,b]$, $\exists f_n$ such that $|f_n(x_n)| > n$. Then by Bolzano-Weierstrauss, we can take $\{x_{n_k}\} < \{x_n\}$ with $x_{n_k} \to x_0$ for some $x_0 \in [a,b]$. So by equicontinuity assumption, we have the following:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall k$ sufficiently large, $|x_{n_k} - x_0| < \delta$, then $n_k < |f_{n_k}(x_{n_k})| \leq |f_{n_k}(x_{n_k}) - f_{n_k}(x_0)| + |f_{n_k}(x_0)| < \varepsilon + \infty$.

1.4 Sep. 4, 2019

Uniform equicontinuity: Let $\varepsilon > 0$, since each f_n is uniformly continuous on $[a,b]$, let $\forall n$, $\delta_n(\varepsilon) = \min\{1, \sup\{\tilde{\delta_n} > 0 | \forall x, y \in [a,b], |x-y| < \tilde{\delta_n} \implies |f_n(x) - f_n(y)| < \varepsilon\}\}$. Our goal is to show that $\delta(\varepsilon) = \inf_n \delta_n(\varepsilon) > 0$. Because then: $\forall 0 \leq x, y \leq b, |x-y| < \delta(\varepsilon) \leq \delta_n(\varepsilon)$, therefore, $f_n(x) - f_n(y)| < \varepsilon, \forall n$. Therefore we have equicontituity.

Argue by contradiction and suppose that $\delta(\varepsilon) = 0$, then $\forall k \geq 1, \exists n_k > 0, \delta_{n_k}(\varepsilon) < 1/k$, and also $\exists a \leq x_k, y_k \leq b$ such that $\delta_{n_k} < |x_k - y_k| < 1/k$, and so $|f_{n_k}(x_k) - f_{n_k}(y_k)| \geq \varepsilon$. Again by Bolzano-Weierstrauss, we can pick a subsequence $\{x_{n_k}\} < \{x_k\}$ such that $x_{k_l} \to^{l \to \infty} x_0$ for some $x_0 \in [a,b]$. And furthermore, $y_{k_l} \to^{l \to \infty} x_0$, since $|y_{k_l} - x_0| \leq |y_{k_l} - x_{k_l}| + |x_{k_l} - x_0| \leq 1/k$. From the assumption of equicontinuous of f_n's at x_0, we have $\exists \delta_1 > 0, \forall l$ sufficiently large so that $|x_{k_l} - x_0| < \delta_1$ and $|y_{k_l} - x_0| < \delta_1$. Then $|f_{n_{k_l}}(x_{k_l}) - f_{n_{k_l}}(x_0)| < \varepsilon/2$ and $|f_{n_{k_l}}(y_{k_l}) - f_{n_{k_l}}(x_0)| < \varepsilon/2$. Now we have $\varepsilon < |f_{n_k}(x_k) - f_{n_k}(y_k)| \leq |f_{n_{k_l}}(x_{k_l}) - f_{n_{k_l}}(y_{k_l})| < \varepsilon$. Contradiction.

Uniform converging subsequence of $\{f_n\}$: First, we construct a subsequence of $\{f_n\}$ converging on a countable dense subset of $[a,b]$. (using uniform bounded of $\{f_n\}$) And must show that the subsequence converges in fact on all of $[a,b]$. (using uniform equicontinuity of $\{f_n\}$). First enumerate all rationals in $[a,b]$: $x_1, \ldots$. And by using boundedness of f_n: $\forall k, \exists M_k > 0, |f_n(x_k)| \leq M_k, \forall n$. And by Bolzano-Weierstrauss, $\forall k$, there exists a subsequence of $\{f_n(x_k)\}$ that converges. Now, pick first a subsequence $\{f_{1,l}\}$ of $\{f_n\}$ that converges at x_1. Next, pick a subsequence $f_{2,l}$ of $\{f_{1,l}\}$ that converges at x_2. Continue this way to get the following table:

$f_{11}, f_{12}, f_{13}, f_{14}, \ldots$
$f_{21}, f_{22}, f_{23}, f_{24}, \ldots$
$f_{31}, f_{32}, f_{33}, f_{34}, \ldots$
$\ldots$

Pick now the diagonal terms to get the following $\{f_{nn}\} < f_n$ that converges at each rational $x_1, \ldots \in [a,b]$. It remains to show that $\{f_{kk}\}$ converges everywhere on $[a,b]$.

1.5 Sep. 6. 2019

Given $\delta > 0$ we can pick $r(\delta) > 0$ and the rationals $x_1, \ldots, n_{r(\delta)}$ in $[a,b]$ such that $\forall x \in [a,b], \exists l \in \{1, \ldots, r(\delta)\}$, such that $|x - x_l| < \delta$. Further, observe that since each $f_{k,k}$ converges at rationals in $[a,b]$, we can also pick $\forall l \in \{1, \ldots, r(\delta)\}, \exists N_l > 0, \forall n, m \geq N_2, |f_{n,n}(x_k) - f_{m,m}(x_l)| < \varepsilon$. Finally, by the triangle inequality, we have: $\exists N = \max\{N_1, \ldots, N_{r(\delta)}\} > 0$,

$\forall n, m \geq N, \forall x \in [a, b], |f_{m,m} - f_{n,n}| \leq |f_{n,n}(x) - f_{m,m}(x_l)| + |f_{n,n}(x_l) - f_{m,m}(x_l)| + |f_{m,m}(x_l) - fm, m(x)| < 3\varepsilon$. □

Note that we are working on $\mathcal{F} = C([a, b], \mathbb{R})$ which is a infinite dimensional vector space of continuous functions on $[a, b]$ with values in $\mathbb{R}$.

Example 1.5.1. 1.

> **Corollary 1.5.2.** *$\{f_k\}$ is continuous and differentiable on $[a, b]$ and $\exists c > 0, \forall k, |f_k(x)|$, $|f_k'(x)| \leq c, \forall x \in [a, b]$, then $\{f_k\}$ has a uniformly converging subsequence.*
>
> *Proof.* Need to check that $\{f_k\}$ is pointwise bounded on $[a, b]$, and it is equicontinuous on $[a, b]$. The first is clear from assumptions. The second follows from mean value theorem. Now apply Ascoli theorem. □

2. $\{f_n(x) = x^n, 0 \leq x \leq 1, n = 1, 2, 3, \ldots\}$ is uniformly bounded but not equicontinuous on $[0, 1]$
3. $\{f_n(x) = x^n, 0 \leq x \leq 1, n = 1, 2, 3, \ldots\}$ is uniformly bounded but not equicontinuous on $[0, 1]$
4. $\{f_n(x) = n, 0 \leq x \leq 1, n = 1, 2, 3 \ldots\}$ is uniformly equicontinuous but not pointwise bounded on $[0, 1]$.

1.6 Sep. 9, 2019

For more special cases of the limit interchange
1. Intercange of limit and the Riemann intergral

Theorem 1.6.1. *$\{f_n\}$ are Riemann integrable on $[a, b]$ (in particular, each f_n is bounded on $[a, b]$). Let $f_n \to f$ uniformly on $[a, b]$, then f is also Riemann Integrable on $[a, b]$ and $\lim_n \int_a^b f_n = \int_a^b \lim_n f_n$.*

Proof. First we show that f is Riemann integrable on $[a, b]$. For each n we define: $\varepsilon_n = \sup |f_n(x) - f(x)| = \|f_n - f\|_{\text{sup}}$. Then $f_n - \varepsilon_n \leq f \leq f_n + \varepsilon$. Thus f is bounded on $[a, b]$. $\forall n, \int_a^b (f_n - \varepsilon_n) = \underline{\int_a^b} (f_n - \varepsilon_n) \leq \underline{\int_a^b} f \leq \int_a^b f \leq \overline{\int_a^b} f_n + \varepsilon = \int_a^b (f_n + \varepsilon_n)$. Therefore, $0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 2\varepsilon(b - a) \to 0$. Thus f is Riemann integrable on $[a, b]$.

For the passage to the limit, just observe: $|\int_a^b f_n(x)dx - \int_a^b f(x)dx| \leq \int_a^b |f_n(x) - f(x)|dx \leq \varepsilon_n(b - a) \to 0$ □

2. Interchange of limits and differentiation.

Remark 1.6.2. Clearly, uniform convergence of (differentiable) $f_n \to f$, implies nothing about convergence of f_n'. Take e.g. $f_n(x) = \sin(nx)/\sqrt{n}, x \in \mathbb{R}$.

Theorem 1.6.3. *$\{f_n\}$ be differentiable on $[a, b]$, and $\{f'_n\}$ converge uniformly on $[a, b]$ and $\{f_n(x_0)\}$ converges uniformly on $[a, b]$ to some f and we have $f'(x) = \lim_n f'_n(x), \forall x \in [a, b]$.*

Proof. Fix $\varepsilon > 0$, pick $N > 0$, $\forall n, m \geq N$, we have $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$ and $|f'_n(x) - f'_m(x)| < \varepsilon/2$. Then, $\forall m, n \geq N$, $\forall a \leq x \leq b, |f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \leq |x - x_0||f'_n(\xi) - f'_m(\xi)| + \varepsilon/2 \leq \varepsilon$. Thus $\{f_n\}$ are uniformly Cauchy on $[a, b]$ and so converges uniformly to some f.

Next, let $x \in [a, b]$ be arbitrary (but fixed for the rest of the proof). Define $g_n(y) = f_n(y) - f_n(x)/(y - x), x, y \in [a, b]$. By assumption, we have $\lim_{y\to x} g_n(y) = f'_n(x)$. Furthermore, $\forall n, m \geq N, |g_n(y) - g_m(y)| = |(f_n(y) - f_m(y) - (f_n(x) - f_n(y)))/(y - x)| = |f'_n(c) - f'_m(c)| < \varepsilon/2(b - a)$, so that g_n is uniformly Cauchy in $a \leq y \neq x \leq b$, and hence $\{g_n\}$ converges uniformly to some limit function.

Finally, we have $\lim_{y\to x} \lim_{n\to\infty} g_n(y) = \lim_{y\to x} \lim_{n\to\infty} f_n(y) - f_n(x)/(y - x) = \lim_{y\to x} (f(x) - f(y))/(y - x) = \lim_{n\to\infty} \lim_{y\to x} g_n(y) = \lim_n f'_n(x)$ by proposition of interchange limits. □

1.7 Sep. 11, 2019

Corollary 1.7.1. *$\{f_n\}$ are Riemann Integrable on $[a, b]$ and $\sum_{i=1}^{\infty} f_n$ converges uniformly on $[a, b]$, then $\int_a^b \sum_n f_n = \sum_n \int_a^b f_n$*

Corollary 1.7.2. *Assume that $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$ cannot be dropped. Take the sequence $f_n(x) = n$.*

1.7.1 A Digression on Taylor expansions and seires

Notation: $k \in \mathbb{N} = \{0\} \cup \mathbb{Z}^+$, $I \subset \mathbb{R}$ is an interval. $C^k(I; \mathbb{R}, \mathbb{C}) = \{f : I \to \mathbb{R}, \mathbb{C} : f, f', \ldots f^k$ exists and are continuous on $I\}$. In part, $C^1(I) = \{$the space of continuous differentiable f on $I\}$. $C^\infty(I) = \bigcap_{k=0}^{\infty} C^k(I)$, and this is the space of functions of C^∞ or smooth functions.

Proposition 1.7.3. *Taylor expansion: $f \in C^{k+1}(I), x, a \in I \subset \mathbb{R}$, open. Then $f(x) = f(a) + f'(a)(x - a) + \ldots + \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{k!} \int_a^x (x - y)^k f^{(k+1)}(y)dy$*

Proof. Use the Fundamental Theorem of Calculus and induct on k. □

Theorem 1.7.4. *Taylor series: $f \in C^\infty(I), x, a \in I \subset \mathbb{R}$, then we can associate with its infinite Taylor series: $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k$.*

Remark 1.7.5. Typically, we adjust f so that $a = 0$.

Remark 1.7.6. Note that this series is an example of a power series: $\sum_{i=1}^{n} c_n(x - a)^n$.

Definition 1.7.7. A (real valued) function $f(x)$ which has a power series expansion at each point of its domain is called *(Real) analytic*.

Example 1.7.8. Polynomials in x, e^x, $\log x$, trig functions $\sin x, \cos x$

Notation: $C^\omega(I) = \{$the space of (real) analytic functions of $I\}$.
We have $C^\omega(I) \subsetneq C^\infty(I) \subsetneq C^k(I) \subsetneq \cdots C^2(I) \subsetneq C^1(I) \subsetneq C^0(I)$

Example 1.7.9. Take $f(x) = \begin{cases} e^{-1/x^2}, x > 0 \\ 0, x \leq 0 \end{cases}$. Check $f^{(k)}(0) = 0$, but $f \in C^\infty(I)$ but $f \notin C^\omega(I)$

Remark 1.7.10. Using $f(x)$ above, we can construct other useful C^∞ functions, such as the *bump function*: $g(x) = \begin{cases} e^{-1/(x+1)^2} e^{-1/(x-1)^2} e^2, |x| \leq 1 \\ 0, otherwise \end{cases}$

Chapter 2

A First Look at Fourier Series

2.1 Sep. 13, 2019

Definition 2.1.1. Let f be Riemann integrable on $[a, b]$, the Fourier Series of f is a power series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx/l}$, where $l = b - a$ and $\hat{f}(n) = 1/l \int_a^b e^{-2\pi inx/l} f(x)dx$ is called the Fourier coefficient of f.

Remark 2.1.2. Typically, $[a, b] = [-\pi, \pi]$ or $[0, 2\pi]$ (also $[0, 1]$ or $[-1, 1]$). Note that both f and continuous Fourier Series can be viewed as either defined on the unit circle $T^1 = S^1$ or defined on 2π periodic function on $\mathbb{R}$: $f(x) = \tilde{f}(e^{ix})$, so that $f(x) = f(x + 2\pi) = f(\cos x + i \sin x)$ by Euler formula

Goal: Understand relation between f and continuous Fourier Series.

Question: Are two functions whose Fourier Series coincide, necessarily equal (uniqueness)? In what sense does the Fourier Series of f converge to f (if at all) (Consequence)? How do we sum the Fourier Seires of f? (Using partial sums? Or Cesaro or Abel? Or something else?) (Summability)

Fourier noticed that any function $f(x)$ can be represented as a sum: $f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$, which let him to speculate that any function $f(x)$ can be decomposed into $\sum_n a_n \cos nx + \sum_n b_n \sin nx = \sum_n c_n e^{inx}$, where $a_n, b_n, c_n \in \mathbb{C}$.

In fact, suppose that $f(x) = \sum_n c_n e^{inx}, x \in \mathbb{R}$. Then a formal calculation gives the following: $\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \int_{-\pi}^{\pi} \sum_n c_n e^{inx} e^{-ikx}dx$, if $\sum_n c_n$ converges uniformly, then the above is: $\sum_n c_n \int_{-\pi}^{\pi} e^{i(n-k)x}dx = 2\pi c_k$. Thus we see that convergence and summability is crucial.

2.1.1 Standard summability method for Fourier Series

Recall the partial sum of the Fourier Series of f has the following: $s_N f(x) = \sum_{|n|\leq N} \hat{f}(n)e^{inx}, -\pi \leq x \leq \pi$, i.e. $s_N f$ is a trigonometric polynomial.

Question: Does $s_N f \to_{N\to\infty} f$? e.g. piecewise $\forall x \in [-\pi, \pi]$? No, in general, because we can alter f at a point without changing the Fourier coefficients. Uniformly on $[-\pi, \pi]$?

No, in general (of course), but yes if f is sufficiently smooth. In some integral? Yes, in a mean-square sense, (i.e, Lebesgue L^2) even if f is only Riemann integrable.

Remark 2.1.3. If f is Riemann integrable, then the Fourier Seires of f converge to f everywhere except on a subset of $[-\pi, \pi]$ of *Lebesgue Measure zero*

Definition 2.1.4. A set $E \subset \mathbb{R}$ is said to be of *Lebesgue measure zero* if $\forall \varepsilon$, $\exists \{I_k\}$ open intervals, such that $E \subset \bigcup_k I_k$ and $\sum_k |I_k| < \varepsilon$ (Here: $I = (a, b]$ or $[a, b]$, then $|I| = b - a$).

2.2 Sep. 16, 2019

Lemma 2.2.1. *Any countable (finite or infinite) union of sets of Lebesgue measure zero is a set of Lebesgue measure zero.*

Proof. Let $E_1, \ldots, E_n$ are sets of measure zero. Fix $\varepsilon > 0$, given n, pick $\{I_k^n\}$ open interval, and that $E_n \subset \bigcup_{k=1}^{\infty} I_k^n$ and $\sum_{k=1}^{\infty} |I_k^n| < \varepsilon/2^n$. Then $\bigcup \infty_{k=1} E_n \subset \bigcup_{n,k} I_k^n$ and $\sum_{k,n=1}^{\infty} |I_k^n| = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |I_k^n| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |I_k^n| \leq \varepsilon$ by proposition of interchange of limits. ☐

Example 2.2.2.
1. One point subsets of $\mathbb{R}$, i.e. $E = \{x_0\}, x_0 \in \mathbb{R}$ has measure zero. $\forall \varepsilon > 0$, take $I_1 = (x_0 - \varepsilon/2, x_0 + \varepsilon/2)$, $I_2 = \ldots = I_n = \emptyset$.

2. Countable collections of points in $\mathbb{R}$ has measure zero by lemma. In particular, the set $\mathbb{Q}$ is a set of lebesgue measure zero in $\mathbb{R}$. In fact, first enumerate the rationals as $r_1, \ldots$, next given $\varepsilon > 0$, for each n, pick $I_n = (r_n - \varepsilon/2^{n+1}, r_n + \varepsilon/2^{n+1})$, so that $\mathbb{R} \subset \cup_n I_n$ and $\sum_{n=1}^{\infty} |I_n| \leq \varepsilon$.

Remark 2.2.3. Therefore, $\mathbb{Q}$ is measure theoretically small and yet $\mathbb{Q}$ are topologically large, i.e. $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall x, y \in \mathbb{R}$, $\exists r \in \mathbb{Q}$ such that $x < r < y$.

2.2.1 Examples of Fourier Series

Example 2.2.4. Let $f(x) = x, -\pi \leq x \leq \pi$. Find the associated Fourier Series of f. Compute: if $n \neq 0$, then $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx$. Then we use integral by parts to get $\frac{(-1)^{n+1}}{in}$. Thus the fourier series is $\sum_{n \neq 0} \frac{(-1)^{n+1}}{in} e^{inx} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2in} (e^{inx} - e^{-inx}) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$. Thus, Fourier series of an odd $f(x) = x$ is a sine series. And this series converges by Leibnitz.

2.3 Sep. 18, 2019

Theorem 2.3.1. *f is Riemann integrable on $[-\pi, \pi]$, and $\hat{f} = 0, \forall f \in \mathbb{Z}$. Then $f = 0$ at all points of continuity.*

Proof. By contradiction. First, assume that f is real valued and WLOG, $x_0 = 0$, is a point of continuity for f and $f(0) > 0$. We will construct a sequence of trignometric polynomials that peak at $x_0 = 0$ as follows:

(1)Since f is continuous at $x_0 = 0$, pick $0 < \delta \leq \pi/2$ so that $f(x) > \frac{1}{2}f(0), \forall |x| > \delta$.

(2)Set $p(x) = \varepsilon + \cos x$, where $\varepsilon > 0$ is chosen so that $|p(x)| \leq 1 - \varepsilon/2, \forall \delta < |x| \leq \pi$

(3)Since $|p(0)| = 1 + \varepsilon$, by continuity, pick $0 < \sigma < \delta$ such that $p(x) \geq 1 + \varepsilon/2, \forall |x| < \sigma$

Next observe that by assumption, we have $\int_{-\pi}^{\pi} f(x)t(x)dx = 0, \forall t(x)$-trigonometric polynomials on $[-\pi, \pi]$. Now, compute $\int_{-\pi}^{\pi} f(x)(p(x))^k dx = (\int_{|x|<\sigma} + \int_{\sigma \leq |x| < \delta} + \int_{\delta \leq |x| \leq \pi}) f(x)(p(x))^k dx$.

Then

$$\int_{|x|<\sigma} f(x)(p(x))^k dx \geq \frac{1}{2} f(0)(1 + \varepsilon/2)^k 2\sigma > 0 \to \infty$$

$$\int_{\sigma \leq |x| < \delta} f(x)(p(x))^k dx \geq 0$$

$$|\int_{\delta \leq |x| \leq \pi} f(x)(p(x))^k dx| \leq \int_{\delta \leq |x| \leq \pi} |f(x)(p(x))^k| dx \leq (1 - \varepsilon)^k \|f\|_{\sup} 2\pi \to 0$$

Thus a contradiction for sufficiently large k. □

Next, assume that f is $\mathbb{C}$ valued. Then, let $u(x) = \frac{f(x)+\bar{f(x)}}{2}$, $v(x) = \frac{f(x)-\bar{f(x)}}{2i}$. Then observe that $\hat{f}(n) = 0, \forall n \in \mathbb{Z}$, then $\hat{u(n)} = \hat{v}(n) = 0$. By the first part, u, v vanish at points of continuity which implies $f = u + iv$ also vanishes at points of continuity.

Goal: If f, g are 2π-periodic and continuous, and $\hat{f}(n) = \hat{g}(n), \forall n \in \mathbb{Z}$ then $f = g$

2.4 Sep. 20, 2019

Theorem 2.4.1 (Convergence Theorem)**.** *f is 2π periodic, continuous and $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$ implies $s_n f \to f$ uniformly on $[-\pi, \pi]$.*

Proof. Observe that $\forall N > M$, $|s_N f(x) - s_M f(x)| = |\sum_{N=M+1}^{M} \hat{f}(n)e^{inx}| \leq \sum_{N=M+1}^{N} |\hat{f}(n)| \to 0$. So that $s_N f$ converges to $g = \lim_{N\to\infty} s_N f$ uniformly on $[-\pi, \pi]$. Futhermore: $\hat{g}(n) = 1/2\pi \int_{-\pi}^{\pi} e^{-inx} g(x)dx = 1/2\pi \int_{-\pi}^{\pi} e^{-inx} \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx} dx = \sum_{k\in\mathbb{Z}} \hat{f}(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-n)x} dx = \hat{f}(n)$. Since g is also continuous on $[-\pi, \pi]$ as a uniform limit of continuous function, the theorem follows from uniqueness theorem. □

Remark 2.4.2. Landau Symbols: ("Big Oh" and "little oh") $f(x) \sim O(g(x))$ as $x \to a$ if $\exists C > 0$ such that $|f(x)| \leq C|g(x)|$ as $x \to a$. In particular, $f(x) \sim O(1)$, this means f is bounded near $x = a$

$f(x) \sim o(g(x))$ as $x \to a$, if $f(x) \sim O(g(x))$ as $x \to a$ and $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$

Theorem 2.4.3 (Decay of Fourier Coefficients)**.** *f is 2π periodic and of class C^k $(k \geq 2)$, then $\hat{f}(x) \sim O(|n|^{-k})$ as $|n| \to \pm\infty$*

Proof. Integrate by parts: $\forall n \neq 0$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx = \frac{1}{2\pi}\frac{1}{-in}e^{-inx}f(x)|_{-\pi}^{\pi} + \frac{1}{2\pi}\frac{1}{in}\int_{-\pi}^{\pi} e^{-inx} f(x)dx = \frac{1}{2\pi}\frac{1}{(in)^k}\int_{-\pi}^{\pi} e^{-inx} f^k(x)dx$ from which we have $|\hat{f}(n)| \leq \frac{1}{(in)^k}\|f^k\|_{\sup}$ □

Corollary 2.4.4 (Convergence Theorem)**.** *f is 2π periodic of demension C^k, then $s_N f \to f$ uniformly on $[-\pi, \pi]$.*

Proof. Just note that: $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \leq |\hat{f}(0)| + \sum_{n\neq 0} |\hat{f}(n)| \leq |\hat{f}(0)| + \|f^k\|_{\sup} \sum |n^{-k}| < \infty$ □

Remark 2.4.5. An important formula: $f \in C^k(\mathbb{T})$, then $\hat{f^{(k)}}(n) = (-in)^k \hat{f}(n)$.

2.5 Sep. 23, 2019

Observe that proof we gave boils down for $f \in C^1(\mathbb{T})$ since $\hat{f}(n) \sim O(\frac{1}{n})$. Therefore the decay as $|n| \to +\infty$. And we are lead to the *harmonic series*: $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \sim \sum_{n\in\mathbb{Z}} \frac{1}{|n|}$. But the result of uniform convergence of $s_N f$ still holds, but requires a more sophisticated techniques. In fact, even integer results holds.

Definition 2.5.1. We say that a function f satisfies a *Holder condition* of class α with $0 \leq \alpha \leq 1$, if $|f(x+h) - f(x)| \leq C|h|^\alpha$ for some constant $C > 0$, and all x, h for such the Left Hand side makes sense. When $\alpha = 1$, we alternatively say that f satisfies *Lipschitz condition* with constant C.

Notation: Let $I \subseteq \mathbb{R}$, $0 < \alpha < 1$, $k \in \mathbb{N}$. $C^\alpha(I)$ =the space of function of Holder class $0 < \alpha < 1$ on I. And $C^{k+\alpha}(I) = C^{k,\alpha}(I)$ = the space of function of class C^k, where k-th derivative is of Holder class $0 < \alpha < 1$ on I. $Lip(I) = C^{0,1}(I)$.

Observe that

$$C^1(I) \subsetneq Lip(I) \subsetneq C^\alpha(I) \subsetneq C^\beta(I) \subsetneq C(I)$$

It turns out that our last corollary holds for $f \in C^\alpha(\mathbb{T})$ on the other hand with $\alpha > 1/2$. $\exists f \in C(\mathbb{T})$ where Fourier Series diverges at a point in $\mathbb{T}$ or even at countably many points in $\mathbb{T}$

2.6 Sep. 25, 2019

Question: How fast/slow does the Fourier Series of f converge if f has "low regularity"?

Example 2.6.1. Example of String plucking (I chose not to copy this example): the upshot is that the estimate on the convergence of the Fourier Series of $f(x) = \pi/2 - |x|$ is sharp. But it is slow, mainly $O(1/N)$.

Example 2.6.2. The Holder Case: the decay of the Fourier Coefficients. Let f be $C^\alpha(\mathbb{T}), 0 < \alpha \leq 1$. First, compute $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} f(n)dx = \frac{-1}{2\pi}\int_{-\pi}^{\pi} e^{-in(x-\frac{\pi}{n}} f(x)dx$. If f is Riemann integrable, 2π periodic, $\int_{-\pi-a}^{\pi+a} f(x)dx = \int_{-\pi}^{\pi} f(x)dx$. Next, $\hat{f}(n) = \frac{1}{2}\hat{f}(n) + \frac{1}{2}\hat{f}(n) = |\frac{1}{4\pi}\int_{-\pi}^{\pi} e^{inx}(f(x) - f(x+\pi/n))dx| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x) - f(x+\frac{\pi}{n})|dx \leq \frac{c}{2}\pi^\alpha \frac{1}{|n|^\alpha} \leq C|\pi/n|^\alpha$

2.7 Sep. 27, 2019

Remark 2.7.1. We have $f \in C^k(\mathbb{T})$, then $\hat{f}(n) \sim O(1/|n|^k)$. And $f \in C^\alpha(\mathbb{T})$, then $\hat{f}(n) \sim O(1/|n|^\alpha)$.

The decay estimate is sharp in the following sense: (Weierstrauss Hardy function) $f(x) = \sum_{k=0}^\infty 2^{-k\alpha} e^{i2^k x}$. And one can show that $f \in C^\alpha$ and $\hat{f}(n = 2^k) = 1/|n|^\alpha$.

2.7.1 Convolution

Definition 2.7.2. Let f, g are 2π periodic and Riemann integrable. Then their *convolution* is defined as: $f \star g(x) = \int_{-\pi}^{\pi} f(x-t)g(t)dt$. Note that the integral is well-defined because product of Riemann Integrable (bounded) are Riemann Integrable.

Proposition 2.7.3 (Basic Properties of convolution)**.** *Let f, g, h be 2π periodic and Riemann Integrable. Then*

1. $f \star (g \star h) = f \star g + f \star h$; $(cf) \star g = c(f \star g) = f \star (cg)$
2. $f \star g = g \star f$
3. $(f \star g) \star h = f \star (g \star h)$
4. $f \star g$ *is a continuous function*
5. $f \hat{\star} g = \hat{f}\hat{g}$
6. *If in addition,* $f \in C^k$, *then* $(f \star g)^{(k)} = f^{(k)} \star g$.

Proof. 1 follows at once from the definition of convolution and the linearity properties of the Riemann Integral.

The 2,3,4,5 is proved by the following strategy. Step 1: proving assuming f, g, h are continuous. Step 2: approximate each f, g, h by a sequence of continuous functions. f_k, g_k, h_k of continuous functions using the following lemma:

Lemma 2.7.4. *If f is Riemann integrable on $[-\pi, \pi]$, then $\exists f_k \in C([-\pi, \pi])$ such that $\forall k, \|f_k\|_{\sup} \leq \|f\|_{\sup}$ and $\int_{-\pi}^{\pi} |f_k(x) - f(x)|dx \to 0$*

And prove it for the limiting function.

4: Assume f, g are continuous. Compute: $\forall -\pi \leq x, y \leq \pi, |f \star g(y) - f \star g(x)| = |\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(y-t) - f(x-t))y(t)dt| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |(f(y-t) - f(x-t))y(t)|dt$ Then by uniform continuity, $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [-\pi, \pi], |x - y| < \delta, \implies |f(x) - f(y)| < \frac{\varepsilon}{\|y\|_{\sup}}$, we know that the above integral is less than ε.

Next, assume f, g are Riemann Integrable. Use lemma to pick approximating sequences f_k, g_k of continuous functions. Write: $f \star g - f_k \star g_k = (f - f_k) \star g + f_k \star (g - g_k)$. And observe the following: $\forall x \in [-\pi, \pi], |(f - f_k) \star g(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y) - f_k(x-y)||g(y)|dy \leq \varepsilon$. Thus, we have shown that $f_k \star (g - g_k), (f - f_k) \star g \to 0$ uniformly convergent.

2.8 Sep. 30, 2019

(v) First assume f, g are continuous. Compute $\forall n$, $\hat{f \star g} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy)dx = \hat{f}(n)\hat{g}(n)$.

Next, assume that f, g are Riemann integrable and pick two approximate sequence f_k, g_k of continuous functions. Compute: $|\hat{f_k \star g_k} - \hat{f \star g}(n)| = |\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}(f_k \star g_k - f \star g(x))dx| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_k \star g_k - f \star g|dx$ which converges to 0 uniformly on $[-\pi, \pi]$. Similarly, $\forall n, |\hat{f}_k(n) - \hat{f}(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{f}_k(n) - \hat{f}(n)|dx$ converges to 0. Therefore, $\hat{f_k \star g_k} = \hat{f_k} \star \hat{g_k}$

(iv) Let $f \in C^{k+1}(\mathbb{T})$, compute $\forall -\pi \leq x \neq y \leq \pi$, $\frac{f\star g(y) - f\star g(x)}{y-x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f\star g(y-t) - f\star g(x-t)}{y-x} g(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(e-t)g(t)dt = f' \star g$ by mean value theorem. □

2.9 Oct. 2, 2019

Definition 2.9.1. A family $\{k_n\}$ is called an approximate identity or a summability kernel in $\mathbb{T}$ if

1. $\forall n, \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x)dx = 1$
2. $\exists c > 0, \forall n, \int_{-\pi}^{\pi} |k_n(x)|dx \leq C$
3. $\forall \delta > 0, \int_{\delta \leq |x| \leq \pi} |k_n(x)|dx \to 0$

Theorem 2.9.2. *$\{k_n\}$ is summability on $\mathbb{T}$, f is Riemann integrable on $\mathbb{T}$, then $\lim_{n\to\infty} k \star f(x) = f(x)$ at each point of continuity of f. Furthermore, if f is continuous on $\mathbb{T}$, then the limit is uniform.*

Example 2.9.3.
1. Dirichlet kernel: limited on standard partial sums. Is not a good summability kernel.
2. Fejer kernel: based on "Cesaro means". Good summability kernel.
3. Poisson kernel: based on "Abel means". Good summability kernel.

2.9.1 Dirichlet Kernel

Example 2.9.4. A nonexample: the Dirichelet: Convolution with the Fourier Series, consider the standard martial sum of the Fourier Series of f: $s_N f(x) = \sum_{|n|\leq N} \hat{f}(n)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{|x|\leq N} e^{in(x-y)} f(y)dy = D_N \star f(x)$

Definition 2.9.5. $D_n(x) = \sum_{|k|\leq n} e^{ikx}$ is called the Dirichlet kernel.

Lemma 2.9.6.
1. *$\forall N, \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$*
2. *$\forall N, D_N(x) = \frac{\sin((N+1/2)x)}{\sin x/2}$*
3. *$\int_{-\pi}^{\pi} |D_N(x)|dx$ goes to ∞*

2.10 Oct. 4, 2019

Proof. 1. $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = \frac{1}{2\pi}\sum_{|k|\le n}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi}2\pi = 1$.

2. Expand: $D_N(x) = e^{inx} + e^{i(n-1)x} + \ldots + e^{ix} + 1 + e^{-ix} + \ldots + e^{-inx}$. Compute $e^{xi/2}D_N(x) - e^{-xi/2}D_N(x) = e^{i(N+1/2)x} - e^{-i(N+1/2)x}$. And multiply both sides by $1/2i$ to get 2.

3. $\int_{-\pi}^{\pi}|D_N(x)|dx = \int_{-\pi}^{\pi}|\frac{\sin((N+1/2)x)}{\sin x/2}|dx = 2\int_0^{\pi}|\frac{\sin((N+1/2)x)}{\sin x/2}|dx \ge 4\int_0^{\pi}|\frac{\sin((N+1/2)x)}{x}|dx = 4\int_0^{N\pi+\pi/2}|\frac{\sin(x)}{x}|dx \ge 4\int_0^{N\pi}|\frac{\sin(x)}{x}|dx = 4\sum_{k=1}^{n}|\sin x/x|dx \ge \frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{k}\int_{(k-1)\pi}^{k\pi}|\sin x|dx = \frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{k}\int_0^{\pi}|\sin x|dx = \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k}$ which goes to infinity. □

Remark 2.10.1. In fact, the above proof shows $\int_{-\pi}^{\pi}|D_N(x)|dx \le \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k} \ge \frac{8}{\pi}\int_1^{n+1}\frac{1}{k}dk = \frac{8}{\pi}\log(n+1) \ge \frac{8}{\pi}\log n$. In fact, a more careful estimate gives: $L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(x)|dx = \frac{4}{\pi^2}\log(n) + O(1)$. Where L_N is called the *Lebesgue constant* or the *Lebesgue number*

Remark 2.10.2. The Dirichlet fails to be a summability kernel on $\mathbb{T}$ because of 3.

2.10.1 Fejer/Cesaro kernel

Definition 2.10.3. We say that a series $\sum_{n=0}^{\infty} c_n$ is *Cesaro summable to* σ if $\sigma_N = \frac{s_0+s_1+\ldots+s_{n-1}}{N} \to \sigma$, wehere s_n is the n-th partial sum of $\sum_n c_n$ and σ_n is called the n-th Cesaro means of $\sum_n c_n$

Lemma 2.10.4. *If $\sum_n c_n$ converges to σ, then $\sum_n c_n$ is Cesaro summable to σ, but the converse fails in general.*

Proof. First, assume $\sigma = 0$, let $\varepsilon > 0$ and since $s_n = \sum_{k=0}^{n} c_k \to 0$, pick $N_1 > 0$ such that for all $n \ge N_1, |s_n| \le \varepsilon/2$. Pick $N_2 > N_1$ such that $\forall n \ge N_2, \frac{|s_0|+|s_1|+\ldots+|s_{N-1}|}{n} \le \varepsilon/2$. Then $\forall n \ge N_2, |\sigma_n| = |\frac{s_0+s_1+\ldots+s_{n-1}}{n}| \le \frac{|s_0|+|s_1|+\ldots+|s_{N-1}|}{n} + \frac{|s_{N-1}|+\ldots+|s_{n-1}|}{n} \le \varepsilon$

Remark 2.10.5. In fact, for any sequence $\{a_n\}$, the argument above shows if $a_n \to 0$, then $\frac{a_0+\ldots+a_{n-1}}{n} \to 0$

Next, assume $\sigma \ne 0$, and consider the sequence $a_n = s_n - \sigma$, since $s_n \to \sigma$, implies $a_n \to 0$, then by the remark above, $\frac{a_0+\ldots+a_{n-1}}{n} = \sigma_n - \sigma \to 0$.

Finally, for the converse consider the following series $\sum_{n=0}^{\infty}(-1)^n$ □

2.11 Oct. 9, 2019

Definition 2.11.1. $\sigma_N f(x) = \frac{s_0 f(x)+\ldots+s_{N-1}f(x)}{N} = F_N \star f(x)$, for $-\pi \le x \le \pi$. For f 2π periodic, Riemann integrable and $s_k f(x)$ is the k the partial sum of the Fourier Series of f. Where $F_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} D_k(x)$ is the Fejer kernel on $\mathbb{T}$.

Lemma 2.11.2. $F_N(x) = \frac{1}{N}\frac{\sin^2 Nx/2}{\sin^2 x/2}$ *and F_N is a summability kernel on $\mathbb{T}$*

Proof. $F_N(x) = \frac{1}{N}\frac{\sin(N+1/2)x}{\sin x/2}\frac{\sin x/2}{\sin x/2} = \frac{1}{N}\frac{1/2}{\sin^2 x/2}(1-\cos x+\cos x-\ldots+\cos(N-1)x-\cos Nx$ where we get our result.

$\forall n, \frac{1}{2\pi N}\int_{-\pi}^{\pi} F_N(x)dx = \frac{1}{2\pi N}\int_{-\pi}^{\pi}\sum D_N(x)dx = \frac{1}{N}\sum\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$

The next follows at once from (1)

The third evaluates to be less than or equal to $\frac{1}{c_\delta}\frac{1}{N}2\pi \to 0$ □

Theorem 2.11.3 (Frejer). *If f is 2π periodic and Riemann integrable. Then $\frac{1}{2\pi}F_N\star f(x) \to f(x)$ at each point of continuity of f: in particular, if $f \in C(\mathbb{T})$ then the Fourier Series of f is uniformly Cesaro-summable to f on $\mathbb{T}$.*

Proof. Follows at once from lemma and abstract kernel theorem. □

Corollary 2.11.4 (Uniqueness of Fourier Series). *f is 2π periodic and Riemann integrable and $\forall n \in \mathbb{Z}, \hat{f}(n) = 0$. Then $f = 0$ at each point of continuity of f.*

Remark 2.11.5. Fejer doesn't say that a continuous $f \in C(\mathbb{T})$ can be recovered from its Fourier series using the standard partial-sum summation. In fact, this is false, we will show it later.

Corollary 2.11.6 (Weierstrass Approximation Theorem 1). *Any continuous function on $\mathbb{T}$ can be uniformly approximated by trigonometric polynomials. I.e. $\forall\varepsilon > 0, \forall f \in C(\mathbb{T}), \exists t_\varepsilon$, a trigonometric polynomial such that $|f(x) - t_\varepsilon(x)| < \varepsilon$*

Proof. Follows at once from Fejer, because Cesaro means are trigonometric polynomials. □

2.12 Oct. 11, 2019

Corollary 2.12.1 (Weierstrass Approximation Theorem ii). *Any continuous function on a closed bounded interval $[a, b]$ can be uniformly approximated by polynomial. $\forall f \in C(\mathbb{T}), \forall\varepsilon > 0, \exists p_\varepsilon$ a polynomial, $|f(x) - p_\varepsilon(x)| < \varepsilon, \forall x \in [a, b]$.*

Proof. Reduce to Weierstrass 1. Observe that approximating $f(x)$ by $p(x)$ uniformly on $[a, b]$ is equivalent to approximating $f(x+a)$ by $p(x+a)$ uniformly on $[0, b-a]$, is equivalent to aaproximating $f(\frac{b-a}{2\pi}(x + a))$ by $p(\frac{b-a}{2\pi}(x + a))$ uniformly on $[0, 2\pi]$. Next, note that it suffices to approximate $g(x) = f(\frac{b-a}{2\pi}(x + a)) - \frac{x}{2\pi}(f(b) - f(a))$ by polynomials uniformly on $[0, 2\pi]$, note, furthermore, that g is continuous on $[0, 2\pi]$ and $g(0) = f(a) = g(2\pi)$, then $g \in C(\mathbb{T})$. Apply Weierstrass 1 to approximate g uniformly by trig. polys. And finally, observe that for each k, we can approximate $e^{ikx} = \cos kx + i\sin kx$ uniformly by $[0, 2\pi]$ by Taylor polynomials. □

Remark 2.12.2.

1. An alternative proof of Weierstrass ii is as follows: consider the functions $k_n(x) = c_n(1 - x^2)^n$, $-1 \le x \le 1$. c_n are chosen so that $\int_{-1}^{1} k_n(x)dx = 1$. Check that $\{k_n\}$ is a summability kernel on $[-1, 1]$. Apply the abstract kernel theorem.

2. Both Weierstrass 1 and Weierstrass 2 can be viewed as density results (compare metric spaces later) similarly to the situation of the rationals as a "dense" subset of the set of all the real numbers $\mathbb{R}$.

Another consequence of Fejer and Fourier Series in:

2.13 Oct. 14

(This part of the notes is taken by Yuxin Lin)

Theorem 2.13.1 (Wiestrass)**.** *There exists a continuous but no-where differentiable function*

Proof: consider the function

$$f(x) = \sum_{n=0}^{\infty} 2^{-\alpha n} e^{i2^n x}$$

f $\in C^\alpha \in C$

Lemma 2.13.2. *Let g be a 2π periodic continous function with*

$$\hat{g}(k) = \begin{cases} a_n, & \text{if } k = \pm 2^n \\ 0 & \text{if otherwise} \end{cases} \tag{2.13.1}$$

and g is differentiable at some point $x_0 \in [-\pi, \pi]$, Then there exists a constant $C > 0$ such that for any $n = 1, 2, ...$, $|a_n| \leq Cn2^{-n}$

Here a_n is a random number. The point here is that $\hat{g}(k)$ is 0 for k not a power of 2

Now assuming the lemma, we want to show that if $f(x)$ has an x_0 differentiable then we will arrive at a contradiction. Can check that $f(x)$ has the required property to apply the lemma. Then we have

$$2^{-\alpha n} = |\hat{f}(2^n)| \leq Cn2^{-n}$$

. This is impossible since α is strictly between 0 and 1.

Now we want to prove the lemma. Without loss of generality can assume that $x_0 = 0$ We can assume this because

$$\hat{g}(x - x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x - x_0)dx$$

$$= \frac{1}{2\pi}\int_{-\pi-x_0}^{\pi-x_0} e^{-in(x+x_0)} g(x)dx$$

$$= e^{-inx_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x)dx$$

This shows that the modulus of the Fourier coefficient doesn't change if we move $g(x)$ to the left or to the right.

We can also assume that $g(0) = 0$. The reason is that subtracting a constant doesn't affect Fourier coefficient except for the coefficient at 0 (Can check this). So if $n \neq 0$ then $g - \hat{g(0)}(n) = \hat{g}(n)$. At 0 the assumption doesn't put any restriction on the Fourier coefficient at 0. So can make this reduction.

The next step is to show that there exists $C > 0$ such that $|g(x)| \leq C|x|$ Since g is differentiable at 0 and $g(0) = 0$, we have

$$\lim_{x \to 0} \frac{g(x)}{x}$$

exists. Thus $\forall \epsilon > 0$, there exists $\delta > 0$ such that if $|x| \leq \delta$

$$|\frac{g(x)}{x}| \leq |\frac{g(x)}{x} - g'(0)| + |g'(0)| \leq \epsilon + |g'(0)| = C_1$$

On the other hand, since g is continuous on $[-\pi, \pi]$, for any x such that $0 < \delta < |x| \leq \pi$, $|g(x)| \leq ||g||_{sup} \leq \delta^{-1}||g||_{sup}|x| = C_2|x|$ Now pick $C = max\{C_1, C_2\}$, we have $|g(x)| \leq C|x|$.

So we have

$$\frac{|g(x)|}{\sin^2 \frac{x}{2}} \leq \frac{C|x|}{\frac{4}{\pi^2}|\frac{x}{2}|^2} = C\pi^2 \frac{1}{|x|}$$

for all $-\pi \leq x \leq \pi$

2.14 Oct. 16, 2019

Recall Fejer kernel: $\frac{1}{n}\frac{\sin^2 Nx/2}{\sin^2 x/2} = F_N(x) = D_0(x) + \ldots + D_{N-1}(x)/N$. By assumption on $g(x)$, we have

Proposition 2.14.1.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\pm i2^n x} F_{2^{n-1}}(x) g(x) dx = a_n^F$$

Proof. Compute $e^{i2^n x}F_{2^{n-1}} = e^{i2^n x}\frac{1}{N}(1 + (e^{-ix} + 1 + e^{ix}) + \ldots + (e^{-i(2^{n-1}-1)x} + 1 + \ldots + e^{i(2^{n-1}-1)x})$, then integrate to get $\hat{g}(-2^n) = a_n^-$. Similarly one can prove the positive sign case. □

Finally, we are ready to prove lemma. Compute $|a_n^+| = |\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i2^n x}F_{2^n-1}(x)g(x)dx| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|F_{2^n-1}(x)||g(x)|dx$ Then by our derived formula, this is $\leq \frac{1}{2\pi}C\pi^2 2^{-(n-1)}2\int_0^{\pi}\frac{\sin^2 2^{n-1}x/2}{x}dx$, we integrate from 0 to $\pi/2^{n-1}$ and from that to π. By using $|\sin x| \leq |x|$ on the first equation and $|\sin x| \leq 1$ on the second, we get $\leq C\pi^3 2^{-n}/4 + 2C\pi 2^{-n}\log 2^{n-1} \cong Cn2^{-n}$.

Remark 2.14.2. A Fourier series that "skips" terms like $\sum a_n e^{i2^n x}$ is called a lacunary Fourier Series.

With some extra work, one can show that $\exists$ functions that are of Holder class $C^\alpha(0 < \alpha < 1)$ that are nowhere differentiable.

The original construction of a nowhere differentiable function by Bolzano was roughly as follows: Construct a sequence of continuous functions with $|x|$-type singularities that pile up as $n \to \infty$. One draw back is that there is no clearly reasonable explicit function for the limit function. Brown observed that motion of tiny particles in a cup of tea turn out paths that similarly have no calculatable velocities.

2.15 Oct. 18, 2019

2.15.1 Abel means/Poison kernel

Definition 2.15.1. A series $\sum_{n>0}^{\infty} c_n$ is *Abel summable to* s if the series $\alpha_r = \sum_{n=0}^{\infty} r^n c_m$ converges $\forall 0 < r < 1$ and $\lim_{r\to 1} \alpha_r = s$. α_r is called the *Abel means* of $\sum_n c_n$

Lemma 2.15.2. *If $\sum_n c_n$ is Cesaro summable to s, hen $\sum_m c_m$ is Abel summable to s but the converse if not true in general.*

Remark 2.15.3. A series converges then the series is Cesaro summable, then the series is Abel summable, but not the other way around

Back to Fourier Series, let f be 2π-periodic and Riemann integrable, then consider the series $\alpha_r f = \sum_{n\in\mathbb{Z}} r^{|n|} \hat{f}(n) e^{inx}$, where $0 < r < 1$ and $-\pi \le x \le \pi$. Observe that $\forall -\pi \le x \le \pi, |\alpha_r f(x)| \le \sum_{n\in\mathbb{Z}} r^{|n|} |\hat{f}(n)| \le C \sum_{n\in\mathbb{Z}} r^{|n|} < \infty$. Then, we have $\alpha_r f(x) = \sum_{n\in\mathbb{Z}} r^{|n|} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} f(y) dy = \frac{1}{2\pi} P_r \star f(x)$

Definition 2.15.4. $P_r(x) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{inx}$ is the *Poisson kernel in* $\mathbb{T}$.

Lemma 2.15.5. $P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$*, and P_r is a summability kernel.*

2.16 Oct. 28, 2019

2.16.1 Quick preview

Problems: Find the solution of the initial value problem $\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}$ and $u(0,x) = f(x)$

Solution: based on the method of separation of variables. We look for solution of the form $u(t,x) = v(t)w(x)$. Substituting gives $v'(t)/v(t) = w''(x)/w(x) = \lambda$ contant independent of both t, x. So that we obtain a system of 2 ODE's, namely $v' - \lambda v = 0$ and $w'' - \lambda w = 0$. Solution for the second equation: an educated guess, look for solution of the form $w(x) = e^{sx}, s \in \mathbb{C}$. We substitute it, so we get $(s^2 - \lambda)e^{sx} = 0$. Since $\lambda = -4\pi^2 n^2$, $w(x) = e^{2\pi inx}$. Similarly $v(t) = e^{-4\pi^2 n^2 t}$. By the superposition principle, note that our equation is linear. $u(t,x) = \sum_{n\in\mathbb{Z}} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}$, with $f(x) = u(0,x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$, where $c_n = \hat{f}(n) = 1/l \int_0^1 e^{-2\pi i n y} f(y) dy$

Note that the series $u(t,x)$ converges absolutely, provided that the series $\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$ converges. Therefore, we have

$$u(t,x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi i n x} = \int_0^1 \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n (x-y)} f(y) dy = H_t \star f(x)$$

.

Definition 2.16.1. $H_t(x) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$ is called the *heat kernel on* $\mathbb{T}$.

Remark 2.16.2. It can be shown that H_t is a summability kernel on $\mathbb{T}$. In fact, it is related to Gauss-Weiestrass kernel in the high dimensional case.

Chapter 3

Metric Space

3.1 Oct. 30, 2019

Definition 3.1.1. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ (sometimes called the distance function) satisfying:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ $\forall x, y, z \in X$ (The triangle inequality).

If 1 is replace by another condition $d(x, y) \geq 0$ and $d(x, x) = 0, \forall x, y \in X$ then d satisfying the three conditions is called a *psuedo-metric* on X. The pair (X, d) is called a *(pseudo) metric space.*

Example 3.1.2. The Euclidean space $\mathbb{R}^n, \mathbb{C}^n$ with the following metric

$$d(x, y) = (\sum_{k=1}^{n} |x_k - y_k|^2)^{1/2}$$

.

Remark 3.1.3. More generally, we can introduce:

$$\forall 1 \leq p < \infty, d_p(x, y) = (\sum_{k=1}^{n} |x_k - y_k|^p)^1/p$$

$$p = \infty, d_\infty(x, y) = \max_{1 \leq k \leq n} |x_k - y_k|$$

Definition 3.1.4. Let (X, d) be a metric space and $x \in X$. An *open ball at x* of radius $r > 0$ is the at $B(x, r) = \{y \in X, d(y, x) < r\}$

Example 3.1.5 (The discrete metric space)**.** Let X be a nonempty set, and $d(x, y) = 1$ if $x \neq y$ or 0 if $x = y$.

Example 3.1.6 (The space of bounded functions)**.** Let S be a nonempty set and let $B(S) = \{f : S \to \mathbb{C} : \exists C > 0, |f(x)| \leq C, \forall x \in S\}$ with $d_{\sup}(f, y) = \sum_{x \in S} |f(x) - y(x)|$

Example 3.1.7. The space of Riemann integralble function on [a,b] $R([a, b]) = \{f : [a, b] \to \mathbb{C} : f \text{ if Riemann integrable on } [a, b]\}$ with $d_{L^1}(f, g) = \int_a^b |f(x) - g(x)| dx$ called the L^1 metric.

3.2 Nov. 1, 2019

Example 3.2.1 (The induced metric)**.** If (X, d) is a metric space and $Y \subset X$ then $d_Y = d|_{Y \times Y}$ is a metric on Y so that (Y, d_Y) is a metric space.

Example 3.2.2 (The product metric)**.** If $(X, d_x), (Y, d_y)$ be two metric spaces. Then $X \times Y$ is also a metric space.

Definition 3.2.3. (X, d) is a metric space. A set $E \subset X$ is open if $\forall x \in E, \exists r > 0$ such that $B(x, r) \subset E$. A set $F \subset X$ is closed if F^c is open. A set $N \subset X$ is a neighborhood of a point $x \in X$ if N contains an open set in X containing x. A set $N \subset X$ is a neighborhood of a set $E \subset X$ if N is a neighborhood of all point in E.

Remark 3.2.4. In any metric space X, the open ball $B(x, r)$ with $x \in X$ and $r > 0$ is an open set. In any metric space X and $\emptyset$ is both open and closed.

Proposition 3.2.5. *Let (X, d) be a metric space. Arbitrary unions of open sets are open, finite intersection of open sets are open. Arbitrary intersection of closed sets are closed, finite union of closed sets are closed.*

Definition 3.2.6. (X, d) metric space and $E \subset X$. The *Interior of* E is the set $E^0 = intE = \cup_{U \in E} U$, U is open. The *Closure of* E is the set $\bar{E} = clE = \cap_{E \subset F} F$, F is closed.

Remark 3.2.7. If $E = B(x, r)$ with $x \in X$ and $r > 0$ then $E^0 = E$ and $\bar{E}$ is the closed ball at x with radius r.

3.3 Nov. 4, 2019

(X, d) is a metric space.

Definition 3.3.1. A point x is a *limit point* of a set $E \subset X$ if every neighborhood of x contains a point $y \in E$ and $x \neq y$. If $x \in E$ is not a limit point of E, then x is an isolated point of E.

Definition 3.3.2. A set $E \subset X$ is *perfect* if E is closed and each of its points is a limit point of E. A set E is *dense in* X if $\bar{E} = X$. A set $E \subset X$ is *Nowhere dense in* X if $(\bar{E})^0 = \emptyset$

Definition 3.3.3. A metric space X is separable if it contains a countable dense subset.

Example 3.3.4. Euclidean spaces $\mathbb{R}^n, \mathbb{C}^n$ are separable with $D = \mathbb{Q}^n$ or $\mathbb{Q}+i\mathbb{Q}$ its countable dense subset.

Example 3.3.5. The space l^2 of square summable sequences with d_{l^2} metric is separable with $D = \{y = \{y_k\}_k | y_k \in \mathbb{Q}, y_{N+1} = y_{N+2} = \ldots = 0 \text{ for some } N > 0\}$.

Example 3.3.6. A discrete metric space (X, d) is separable if X is countable.

Example 3.3.7. Let S be a nonempty set. The space of bounded functions in $S, B(S)$ with the sup metric, is separable if S is finite.

Example 3.3.8. $C([a, b])$ the space of continuous functions on $[a, b]$ with the sup metric. It is separable with $D =$ all polynomials on $[a, b]$ with rational coefficients. By Weierstrass Approximation theorem.

3.4 Nov. 6, 2019

Example 3.4.1 (The Cantor Set)**.** Consider the middle "thirds" subsets of $[0, 1]$. Let $C_0 = [0, 1], C_1 = [0, 1/3] \cup [2/3, 1]$ etc.

Proposition 3.4.2. *C is closed, nowhere dense, perfect and of measure zero.*

3.4.1 Topological spaces

Definition 3.4.3. Let X be a nonempty set. A *topology* on X is a family τ of subsets such that $\emptyset, X \in \tau$, τ is closed under infinite unions and finite intersections. The pair (X, τ) is a topological space.

Example 3.4.4. X is a nonempty set, $\tau = \{\emptyset, X\}$ trivial topology. $\tau = P(X) = 2^x$ can be seen as the discrete topology.

Example 3.4.5. X infinite set. $\tau = \{U \subset X, U = \emptyset, U^c \text{ is finite}\}$. Cofinite topology

Example 3.4.6. (X, d) metric space. $\tau = \{\text{ all open subset of } X \text{ w.r.t } d\}$. Metric topology

Example 3.4.7. (X, τ) topological space and $Y \subset X$, then $(Y, \tau|_Y)$ is a topological space called the induced topology.

Definition 3.4.8. Let τ_1 and τ_2 are two topologies in X. Then we say that τ_1 is stronger (finer) then τ_2 if $\tau_2 \subset \tau_1$.

Definition 3.4.9. $(X, \tau_1), (X, \tau_2)$ be two metric spaces, then a function $f : X_1 \to X_2$ is *continuous* at $x \in X_1$ if $\forall \varepsilon > 0, \exists \delta > 0, \forall y \in X_1, d_1(y, x) < \delta$ then $d_2(f(y), f(x)) < \varepsilon$. In other words, $\forall \varepsilon > 0, \exists \delta > 0, B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$.

Proposition 3.4.10. *$f : X_1 \to X_2$ is continuous $\iff$ $\forall U \subset X_2$, $f^{-1}(U) \subset X$.*

3.5 Nov. 8, 2019

Definition 3.5.1. $(X_1, d_1), (X_2, d_2)$ metric spaces. A function $f : X_1 \to X_2$ is an isometry if $\forall x, y \in X_1, d_1(x, y) = d_2(f(x), f(y))$. $(X_1, d_1), (X_2, d_2)$ are *isometric* if there is an isometry onto between them.

Remark 3.5.2. An isometry is always "into" but not necessarily "onto". Eg: $x \in \mathbb{R} \to (x, 0) \in \mathbb{R}^2$.

Proposition 3.5.3 (Elementary properties: Separation axioms)**.** *(X, d) metric space,*

1. *Every one point subset of X is closed. (T_1 axiom)*
2. *$\forall x \neq y$, $\exists U, V \subset X, x \in U, y \in V$ with $U \cap V = \emptyset$ (T_2 axiom or Hausdorff)*
3. *$\forall F \subset X$ closed, $\forall x \in F$, $\exists U, V \subset X$ with $U \cap V = \emptyset, F \subset U, x \in V$. ($T_3$ axiom, regular axiom)*
4. *$\forall E, F \subset X$ closed with $E \cap F = \emptyset, \exists U, V \subset X, U \cap V = \emptyset$ such that $E \subset U$, $F \subset V$ (T_4 property, normal property).*
5. *$\forall E, F \subset X$ with $E \cap F = \emptyset$, $\exists f : X \to [0, 1]$ and such that $f|_E = 0$ and $f|_F = 1$ (Urysohn's Lemma)*

Definition 3.5.4. (X, d) metric space, a sequence $\{x_n\} \subset X$ is *Cauchy* if $\forall \varepsilon > 0, \exists N > 0, \forall n, m \geq N, d(x_n, x_m) < \varepsilon$

Definition 3.5.5. $E \subset X$ is *complete* if every Cauchy seqience in E converges to a limit in E.

Proposition 3.5.6. *A closed subset of a complete metric space is complete. A complete subset of any metric space is closed.*

3.6 Nov. 11, 2019

Definition 3.6.1. (X, d) is a metric space, $E \subset X$

1. $\text{diam}(E) = \sup_{x,y \in E} d(x, y)$ is called the diameter of E
2. $E \subset X$ is bounded if $\text{diam}(E) < \infty$
3. $E \subset X$ is totally bounded if $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε

Remark 3.6.2. In any metric space $int(E)$ is totally bounded, then so is $\bar{E}$. In $\mathbb{R}$ Euclidean metric, E is totally bounded iff E is bounded.

Theorem 3.6.3. *(X, d) metric space, $E \subset X$ the following are equivalent:*

1. *E is complete and totally bounded*

2. *Every sequence in E has a converging subsequence in E*

3. *Every covering of E by open sets has a finite subcovering.*

Definition 3.6.4. A set $E \subset X$ satisfying any of the above condition is said to be *compact*.

3.7 Nov. 13, 2019

Remark 3.7.1. Any compact subset of a metric space is closed and bounded. The converse is false in general but true in $\mathbb{R}^n$, in fact, and closed subset of a compact set is compact (in any metric space)

Definition 3.7.2. X nonempty set. Two metrics d_1 and d_2 on X are equivalent if $\exists c_1, c_2 > 0$ $\forall x, y \in X$, $c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)$

Remark 3.7.3. Note that equivalent metrics have the same open sets (any sequences etc.)

Proposition 3.7.4. *A metric space X is compact iff every collection of closed subsets of X with the finite intersection property has a non-empty intersection.*

Definition 3.7.5. (X, d) is a metric space, a collection $\{F_\alpha\}$ of subsets of X has the finite intersection property if every intersection of finitely many of its members is non-empty

Corollary 3.7.6. *If $\{K_m\}$ are non-empty compact subsets of a metric space X with $K_n \supset K_{n+1}$, then $\cap_{n=1}^{\infty} K_n \neq \emptyset$.*

Theorem 3.7.7. *The Cantor set C is compact, and uncountable.*

Proof. $C = \cap_{k=0}^{\infty} C_k$. Closed and bounded in $\mathbb{R}$ and thus is compact. Claim: a non-empty perfect subset of a complete metric space is necessarily uncountable. Pf: Assume for simplicity that the metric space is compact. Let P be aperfect and countable. So we can enumerate $P = \{x_1, \ldots\}$. Construct a sequence of nested closed set as follows: let $B_1 = \{$ open ball at $x_1\}$ so that $B_1 \cap P \neq \emptyset$, and $K_1 = \bar{B}_1 \cap P$. Since P is perfect (so that $x_1 = \lim_{n \neq 1} x_n, x_n \in P$. Let $B_2 = \{$ open ball at $x_{n \neq 1}\}$ so that $B_2 \cap P \neq \emptyset$, with $\bar{B_2} \subset B_1$, and $x_1 \notin \bar{B_2}$. Let $K_2 = \bar{B_2} \cap P$. Continue to get $K_n = \bar{B}_n \cap P$ with $\bar{B}_n \subset B_{n-1}$ and $x_{n-1} \notin \bar{B}_n$. Then K_n are nonempty, compact and nested, and $\cap K_n = \emptyset$. Thus we have a contradiction. ☐

3.8 Nov. 18, 2019

Definition 3.8.1. $f : X \to Y$ is a homeomorphism if f is continuous, one-to-one and onto and f^{-1} is continuous.

Proposition 3.8.2. *X, Y are metric spaces, X is compact, $f : X \to Y$ is continuous. Then*

1. $f(X) \subset Y$

2. f *is uniformly continuous*

3. if $Y = \mathbb{R}$ *then* f *attains its maximal or minimum values*

4. If f *is a bijection, then it is a homeomorphism.*

Theorem 3.8.3 (Baire Category Theorem)**.** (X, d) *complete metric space,*

1. if $\{U_n\}$ *are open and dense in* X*, then* $\cap_n U_n$ *is also dense in* X

2. X *is not a countable union of nowhere dense sets*

Definition 3.8.4. (X, d) is a metric space, $E \subset X$, then E is called a set of first Baire catetory if E is a countable union of nowhere dense sets. Otherwise E is of the second Baire Category.

Remark 3.8.5. 1. Baire category theorem implies that every complete metric psace is of second Baire category

2. Baire theorem can be used as a tool to prove existence results.

3.9 Nov. 20, 2019

Definition 3.9.1. A function f is a contraction if it is Lipchitz with constant $0 < c < 1$. A fixed point in X is a point such that $f(x) = x$

Theorem 3.9.2 (Banach's Contraction Principle)**.** (X, d) *is a complete metric space,* $f : X \to X$ *is a contraction, then* f *has a unique fixed point in* X

Proof. Pick any $x_0 \in X$ and define $x_{n+1} = f(x_n)$. Observe that $\forall n \geq 1$ $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq cd(x_m, x_{m-1}) \leq c^n d(x_1, x_n)$. Further more, we have $\forall n > m$, $d(x_n, x_m) \leq \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \leq \sum_{k=m}^{n-1} c^k d(x_1, x_0) \leq c^m \sum_{k=0}^{\infty} c^k d(x_1, x_0) = \frac{c^m}{1-c} d(x_1, x_0) \to 0$. So that $\{x_n\}$ is Cauchy in $X-$complete and thus $x_n \to x$ for some $x \in X$. Clearly, x is a fixed point of f, if we consider the limit.

We now consider uniqueness: Suppose f has two distinct fixed point. $x = f(x), y = f(y)$, then $d(x, y) = d(f(x), f(y)) \leq cd(x, y) < d(x, y)$. A contradiction. □

Remark 3.9.3. Banach's contraction Principle is one of the main tools in proving existence and uniqueness theorems for ODE's and PDE's.

Chapter 4

Elements of Functional Analysis

4.1 Nov. 20, 2019

Definition 4.1.1. Let X be a linear space over $k = \mathbb{R}$ or $\mathbb{C}$. A *norm* on X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying

1. $\forall x \in X, \|x\| \geq 0$ with $x = 0$ iff $x = 0$.
2. $\forall x \in X, \forall \alpha \in k, \|\alpha x\| = |\alpha| \|x\|$
3. $\forall x, y,\ \|x + y\| \leq \|x\| + \|y\|$

If (1) is replaced with $\forall x \in X, \|x\| \geq 0$, then $\|\cdot\|$ is called a psuedo-norm on X. The pair $(X, \|\cdot\|)$ is called a (psuedo-)normed space.

Remark 4.1.2. Any normed space $(X, \|\cdot\|)$ is a metric space with $\forall x, y \in X, d(x, y) = \|x - y\|$

Definition 4.1.3. A complete normed space is called a *Banach space*.

4.2 Nov. 22, 2019

Example 4.2.1 (Examples of Banach Space)**.**

1. $\mathbb{R}^n, \mathbb{C}^n$ with Euclidean norms are Banach spaces
2. X a nonempty set, $B(x)$ =the space of bounded functions on X with image in $K : \mathbb{R}$ or $\mathbb{C}$, with the p norm, $\|f\|_s = \sup_{s \in X} |f(x)|$
3. (X, d)-metric space, $C_b(X) = B(X) \cap C(X)$ (bounded functions and continuous functions on X), a bounded conitnuous functions on X. Equip with the sup norm.
4. $C^k([a, b])$ =the space of C^k functions on $[a, b]$, with $\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_{\sup}$ is Banach.

5. $C^\alpha([a,b])$ The space of Holder-α continuous function on $[a,b]$, equipped with the (Holder) norm: $\|f\|_\alpha = \|f\|_{\sup} + \sup_{x\neq y, x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$. This is also Banach. This is more technical.

6. l^2: The space of square summable sequences of real and complex numbers, with the norm $\|x\|_{l^2} = \sqrt{\sum_{i=1}^\infty |x_k|^2}$

Example 4.2.2. An importent example: The space of bounded linear operators. Let X, Y be normed space with $k = \mathbb{R}, \mathbb{C}$:

Definition 4.2.3. A linear operator $T : X \to Y$ is a map satisfying $\forall x \in k, \forall x, y \in X, T(ax+y) = aT(x)+T(y)$. $T : X \to Y$ is *bounded* if $\exists c > 0, \forall x \in X, \|T(x)\| \leq c\|x\|$. The *operator norm of* T: $X \to Y$, is the number $\|T\| = \sup_{x\in X, \|x\|\leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \inf\{C > 0 : \|Tx\| \leq C\|x\|, \forall x \in X\}$

Proposition 4.2.4. *X, Y normed space, $T : X \to Y$ linear space, the following are equivalent*

1. *T is uniform continuous on X,*

2. *T is continuous on X,*

3. *T is continuous at $x = 0$*

4. *T is bounded*

Proof. $1 \implies 2 \implies 3$ is obvious.

$3 \implies 4$ Pick $\delta > 0, \forall x \in X, \|x\| < \delta$, then $\|Tx\| \leq 1 = \varepsilon$ so that $\forall x \in X, 1 \geq \|T(\delta \frac{x}{\|x\|})\|$. Therefore the following is equivalent: $\forall x \in X, \|Tx\| \leq \frac{1}{\delta}\|x\|$.

$4 \implies 1$ $\forall x, y \in X$, $\|Tx - Ty\| = \|T(x-y)\| \leq C\|x-y\|$. □

4.3 25 Nov. 2019

Proposition 4.3.1. *X is normed, Y is Banach, then $L(X, Y) = \{T : X \to Y : T$ linear and continuous operator$\}$ with the operator norm $\|T\| = \inf\{C > 0, \|Tx\| \leq C\|x\|\}$ is Banach.*

Remark 4.3.2. Let $X = Y$ be Banach. Note that $\forall T, S \in L(X)$, we have $\forall x \in X, \|ST(x)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$ so that $ST \in L(X)$ (note that linearity of ST is clear)

Definition 4.3.3. A Banach space that is also an algebra such that the norm of the product of its elements is at most the product of the norms is called a *Banach algebra*

Example 4.3.4. $(C_b(x), \|\cdot\|_{\sup})$ is a Banach algebra with pointwise multiplication of functions. $(L(X), \|\cdot\|_{operator}$ is a Banach algebra with composition of linear operators

Definition 4.3.5. let X be a normed space over k, the space $X^* = L(X,k)$ is called the *dual space* of X.

Theorem 4.3.6 (Banach-Steinhaus Theorem)**.** *X Banach, Y is normed. $\{T_k\} \subset L(X,Y)$, if $\sup_{k\in K} \|T_k x\| < \infty, \forall x \in X$, then $\sup_{k\in K} \|T_k\| < \infty$*

Proof. $\forall n$, let $E_n = \{k \in X, \sup_{k\in K} \|T_k x\| \leq n\} = \cap_{k\in K}\{x \in X \|T_k x\| \leq n\}$ which is closed. Thus E_n is closed in X. $\cup_n E_n = X$. By Baire, at least one of E_n cannot be nowhere dense. Thus, $\exists n_0 \exists x_0 \in E_{n_0} = \bar{E_{n_0}}$, $\exists r_0 > 0, B(x_0, r_0) \subset E_{n_0}$. Note: $\forall \|x\| \leq \frac{r_0}{2}, x_0 - x \in x_0 + B(0, \frac{r_0}{2}) = B(x_0, \frac{r_0}{2}) \subset E_{n_0}$. So $\forall \|x\| \leq \frac{r_0}{2}, \forall k \in K$, $\|T_k x\| \leq \|T_k(x - x_0)\| + \|T_k x_0\| \leq n_0$. Then $\forall \|n\| \leq 1, \forall k \in K, \|T_k x\| \leq 2n_0 \frac{2}{r_0} = \frac{4n_0}{r_0}$. Then we take the sup. □

4.4 Dec. 2, 2019

Remark 4.4.1. The same result holds if we assume $\sup_{\alpha\in A} \|T_\alpha x\| < \infty, \forall x \in E \subset X$, where E is of second Baire Category.

Corollary 4.4.2. *If X, Y are Banach, $\{T_n\} \subset L(X,Y)$, $\forall x \in X$, $T(x) = \lim_{n\to\infty} T_n(x)$ exists and $\|T\| \leq \liminf \|T_n\| < \infty$.*

Example 4.4.3 (Simple examples of non-complete metric spaces)**.**

1. $\{f \in C([a,b]) : f(x) > 0, \forall a \leq x \leq b\}$ with the sup-metric
2. $X = B(\bar{0},1) - \{0\} \subset \mathbb{R}^2$ with the Euclidean metric.
3. $X = \{\frac{1}{x} : n = 1, \ldots\} \subset \mathbb{R}$ with the euclidean metric.

4.4.1 Compactness in spaces of continuous functions

Lemma 4.4.4 (Riesz)**.** *X is Banach, $W \subsetneq X$ proper closed subspace. Then $\exists\{x_n\} \in X$ such that $\|x_n\| = 1$ and $d(x_n, W) \to_{n\to\infty} 1$.*

Proof. Let $x \in X - W$ and pick a minimizing sequence $\{w_n\} \subset W$ such that $\|x - w_n\| \to d(x,W) = \inf_{w\in W} \|x - w\|$. Let $x_n = \frac{x - w_n}{\|x - w_n\|}$. Then $\|x_n\| = 1, \forall n = 1, \ldots$ and furthermore, $d(x_n, W) = \inf_{w\in W} \|\frac{x-w_n}{\|x-w_n\|} - w\| = \frac{1}{\|x-w_n\|} \inf_{w\in W} \|x - (w_n + \|x - w_n\| w)\|$. We know that $w_n + \|x - w_n\| w \in W$. Thus we have $\frac{d(x,W)}{\|x-w_n\|}$. Thus $d(x_n, W) \to 1$. □

Theorem 4.4.5. *The closed unit ball in an infinite dimensional Banach space is never compact.*

Proof. Since X is infinite dimensional, pick any increasing sequence of subspaces $W_0 \subsetneq W_1 \subsetneq \ldots \subsetneq W_n \subsetneq \ldots \subsetneq X$. Then by Rietz lemma, $\forall n = 1, 2, \ldots$, pick x_n with $\|x_n\| = 1$ and $d(x_n, W_{n-1}) \geq \frac{1}{2}$ such that $\forall n \neq m, d(x_n, x_m) \geq \frac{1}{2}$. Therefore, we have constructed an infinite sequence $\{x_n\} \subset B_X\bar{(}0,1)$ without converging subsequences. □

Remark 4.4.6. Unlike in finite dimensions, compactness is a rare phenomenon in infinite dimensional spaces. However, there are some useful tools to identify compact sets.

Definition 4.4.7. (X, d) metric space, F is a family of k-valued space X, F is *equicontinuous* at $x \in X$ if $\forall \varepsilon > 0, \exists \delta > 0, \forall y \in X, d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon, \forall f \in F$, F is pointwise bounded on X if $\forall x \in X, \exists M > 0$ such that $|f(x)| \leq M, \forall f \in F$, F is *Uniformly equicontinuous in* X if F is equicontinuous $\forall x \in X$ with $\delta > 0$ independent of x. F is *Uniformly bounded on* X if F is pointwisely bounded $\forall x \in X$ with $M > 0$ independent of x

Theorem 4.4.8 (Arzela-Ascoli)**.** *(X, d) compact metric space, if $\{f_n\}$ is a sequence of k-valued functions on X that are equicontinuous $\forall \in X$ and pointwisely bounded on X. Then $\{f_n\}$ is uniformly equicontinuous on X, $\{f_n\}$ is uniformly bounded on X and $\{f_n\}$ has a uniformly converging subsequences.*

Proof. Only notational changes are need to adapt the proof of Ascoli from (from the beginning of the semester). □

Definition 4.4.9. If $\bar{E}_{\text{sup}}$ is a compact set, then E is called *relatively compact.*

Corollary 4.4.10. *If (X, d) is a compact metric space, $E \subset C_b(X) = C(x)$, this is relatively compact iff E is uniformly bounded and uniformly equicontinuous on X.*

4.5 Dec. 4, 2019

Let (X, d) be a compact metric space. Recall that $C(X, k)$ is a Banach algebra, under pointwise multiplication of functions.

Definition 4.5.1. A set $A \subset C(X, k)$ is a *subalgebra* if A is closed under addition, scalar multiplication and pointwise multiplication of function.

Definition 4.5.2. An subalgebra $A \subset C(X, k)$ is said to *separate points in* X if $\forall x_1 \neq x_2 \in X, \exists f \in A, f(x_1) \neq f(x_2)$

Remark 4.5.3. If $A \subset C(X, k)$ is a subalgebra, then so is $\bar{A}^{\|\cdot\|_{\text{sup}}}$. If $A \subset C(X, k)$ is a subalgebra that is closed under complex conjugation, then so is $\bar{A}^{\|\cdot\|_{\text{sup}}}$.

Theorem 4.5.4 (Stone-Weierstrauss)**.** *(X, d) compact metric space.*

1. *If A is a subalgebra of $C(X, \mathbb{R})$ that separates points in X and contains constant, then A is dense in $C(X, \mathbb{R})$*

2. *If A is a subalgebra of $C(X, \mathbb{C})$ that separates points in X, contains constants and is closed under complex conjugation, then A is dense in $C(X, \mathbb{C})$.*

Proof. Firstly one can show that if $f_1, f_n \in \bar{A}$, then $\max / \min\{f_1, f_n\} \in \bar{A}$. Then we prove (1). Let now $f \in C(X, \mathbb{R})$. The proof consists of two steps,

Step I Claim: $\forall x \in X, \forall \varepsilon > 0, \exists g_x \in \bar{A}$ such that $g_x(x) = f(x)$ and $g_x(z) > f(z) - \varepsilon, \forall z \in X$.

Proof of Claim: fix $x \in X$ and $\varepsilon > 0$. Since A separates points in X, then $\forall y \neq x$, $\exists h_y \in \bar{A}, h_y(x) = f(x)$ and $h_y(x) = f(y)$. Because, first, pick some $\tilde{h} \in A$ with $\tilde{h}(x) \neq \tilde{h}(y)$. Next, pick $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta\tilde{h}(x) = f(x), \alpha + \beta\tilde{h}(y) = f(y)$ and now set $h_y() = \alpha + \beta\tilde{h}() \in \bar{A}$. If $y = x$, then set $h_2() = f(x) \in \bar{A}$.

4.6 Dec. 6, 2019

Now, we have: $\forall y \in X, \exists U_y \subset X$ open, such that $h_y(z) > f(z) - \varepsilon, \forall z \in U_y$. Cover $X = \cup U_y$. By compactness of X, pick a finite subcover $U_{y_1}, \ldots, U_{y_n}$ and define $g_x = \max\{h_{y_1}, \ldots, h_{y_n}\}$. Then $g_x \in \bar{A}$ by preliminary result, and $g_x(x) = f(X)$ by construction, and $\forall z \in X = \cup U_y$, $g_x(z) > f(z) - \varepsilon$

Step 2: Using claim, we now also have $\forall x \in X, \exists V_x \subset X$ open $g_x(z) > f(z) + \varepsilon, \forall z \in V_x$. One can prove this as we prove the previous step. So that we have shown the following $\forall f \in C(X, \mathbb{R}), \forall \varepsilon > 0, \exists h \in \bar{A}, \|h - f\|_{\sup} < \varepsilon$.

Consider (2), let $A_{\mathbb{R}} = A$ be the functions in A that have values in $\mathbb{R}$. Clearly $A_{\mathbb{R}}$ is closed under addition and multiplication by scalars in $\mathbb{R}$, pointwise products and contains constants. So that $A_{\mathbb{R}}$ is a subalgebra of $C(X, \mathbb{R})$ itself contains $\mathbb{R}$. If $f = u + iv$, then $\bar{f} \in A$ provided that $f \in A$. By assumption, A is closed under conjugation. Therefore, $u = \frac{1}{2}(f + \bar{f}), v = \frac{1}{2}(f - \bar{f}) \in A$. Since A separate points in X. $\forall x, y \in X, \exists f = u + iv \in A$ such that $f(x) \neq f(y)$. Then we use part (1). □

4.7 Dec. 9, 2019

4.7.1 Hilbert Spaces

Definition 4.7.1. Let V be a metric space over $\mathbb{C}$. An inner product on V is a function $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that

1. $\alpha x + y, z) = \alpha(x, z) + (y, z)$
2. $(x, y) = (\bar{y, x})$

The pair $(V, (,))$ is called an inner-product (pre-Hilbert) space.

Remark 4.7.2. The number $\|x\| = \sqrt{(x, x)}$ defines a norm on V

Definition 4.7.3. If the topology defined by $\|\cdot\|$ above is complete, then $(V, (,))$ is called a Hilbert space.

Example 4.7.4. $\mathbb{R}^n, \mathbb{C}^n$ with the standard euclidean inner product $(x, y) = \sum_{i=1}^n x_i \bar{y}_i$ are Hilbert spaces

Definition 4.7.5. Let $(V, (,))$ be an inner product space. We say that $x, y \in V$ are orthogonal if $(x, y) = 0$. We notate it as $x \perp y$. We say that $\{x_k\} \subset V$ is orthogonal if $x_k \perp x_j \forall k \neq j$. We say that $\{x_k\} \subset V$ is orthonormal if in addition $\forall k, \|x_k\| = 1$.

Proposition 4.7.6. *Let $(V, (,))$ be a inner-product space.*

1. $\forall x_1, \ldots, x_n \in V, x_k \perp x_j, \forall k \neq j$, *then* $\| \sum x_k \|^2 = \sum \|x_k\|^2$ *(Pythagoras)*
2. $\forall x, y \in V, |(x, y)| \leq \|x\| \|y\|$ *(Schwarz, aka, Cauchy Schwarz)*
3. $\forall x, y \in V, \|x + y\| \leq \|x\| + \|y\|$ *(Triangle inequality).*
4. $\forall x, y \in V, \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ *(Parallelogram law)*

4.8 Dec. 11, 2019

Example 4.8.1. $l^2(\mathbb{N}) = \{x = \{x_n\} | x_k \in \mathbb{C}, \sum |x_k|^2 < \infty\}$. The space of square summable sequences of complex numbers. Clearly, $l^2(\mathbb{N})$ is a vector space over $\mathbb{C}$. Define an inner produce by $(x, y) = \sum x_k \bar{y}_k$. Therefore, $(l^2(\mathbb{N}), \| \cdot \|_{l^2}$ is Banach, hence complete, hence Hilbert.

Example 4.8.2. An important non-example: $R([a, b]) = \{$the space of complex valued Riemann-integrable functions on $[a, b]\}$. Clearly, $R([a, b])$ is a vector space over $\mathbb{C}$ (under pointwise operation). Define an inner product by $(f, g)_{L^2} = \int_a^b f(x) g(\bar{x}) dx$, with the coresponding L^2 norm. $\|f\|_{L^2} = (\int_a^b |f(X)|^2 dx)^{1/2}$.

Remark 4.8.3. Notive that $\| \cdot \|_{L^2}$ is not non-degenerate on $R([a, b])$. Nevertheless, it is easy to remedy this by passing from $R([a, b])$ to space $\tilde{R}([a, b])$:Riemann integrable functions which are equal everywhere on $[a, b]$ except on a set of measure zero.

Proposition 4.8.4. $(R([a, b], (,)_{L^2})$ *is not complete.*

Remark 4.8.5. This is one reason for replacing the Riemann integral with a better integral tool, the Lebesgue integral.