Structure Theorem for Artin Rings

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Theorem 0.1 (Structure Theorem for Artin Rings). *An Artin ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings*

Proof. Let $\mathfrak{m}_i (1 \leq i \leq n)$ be the distinct maximal ideals of A. Then $\prod_{i=1}^n \mathfrak{m}_i^k \subseteq (\cap_{i=1}^n \mathfrak{m}_i)^k = \mathfrak{R}^k = 0$ by Proposition 8.4. Then $\prod_{i=1}^n \mathfrak{m}_i^k = 0$ for some $k > 0$. By Proposition 1.16, the ideal $\mathfrak{m}_i^k$ are coprime in pairs, by proposition 1.10, $\prod_{i=1}^n \mathfrak{m}_i^k = (\cap_{i=1}^n \mathfrak{m}_i)^k$. Then by Proposition 1.10 again, the natural mapping $\phi : A \to \prod(A/\mathfrak{m}_i^k)$ is both injective (since $(\cap_{i=1}^n \mathfrak{m}_i)^k$ is 0) and surjective (since the ideal $\mathfrak{m}_i^k$ are coprime in pairs). Thus ϕ is an isomorphism. Each $A/\mathfrak{m}_i^k$ is an Artin local ring. Hence, A is a direct product of Artin local rings.

Conversely, assume $A \cong \prod_{i=1}^m A_i$, where A_i are Artin local rings. Then for each i we have natural surjective homomorphisms as projections $\phi_i : A \to A_i$. Let $\mathfrak{a}_i = \ker(\phi_i)$, then by Proposition 1.10, $\mathfrak{a}_i$ are pairwise coprime, and $\cap \mathfrak{a}_i = 0$. Let $\mathfrak{q}_\mathfrak{i}$ be the unique prime ideal of A_i, $\mathfrak{p}_i$ be its contraction $\phi_i^{-1}(\mathfrak{q}_i)$. The ideal $\mathfrak{p}_i$ is prime and is maximal by proposition 8.1. Since $\mathfrak{q}_i$ is nilpotent, then $\mathfrak{a}_i$ is $\mathfrak{p}_i$-primary, and hence $\cap \mathfrak{a}_i = (0)$ is a prime decomposition of the zero ideal in A. Since $\mathfrak{a}_i$ are pairewise coprime, so are $\mathfrak{p}_i$, and they are therefore isolated prime ideals of (0). Hence all the primary components $\mathfrak{a}_i$ are isolated, and therefore uniquely determined by A by the second uniqueness theorem (4.11). Hence $A_i \cong A/\mathfrak{a}_i$ are uniquely determined by A. □

Example 0.2. A ring with only one prime ideal need not be Noetherian (Hence not an Artin ring by Theorem 8.5). Let $A = k[x_1, x_2, \ldots]$ be the polynomial ring in a countably infinite set of indeterminates x_n over a field k. Let $\mathfrak{a}$ be the ideal $(x_1, x_2^2, \ldots, x_n^n, \ldots)$. The ring $B = A/\mathfrak{a}$ has only one prime ideal, namely the image of $(x_1, x_2, \ldots, x_n, \ldots)$. Hence B is a local ring of dimension 0, but B is not Noetherian.

If A is a local ring, $\mathfrak{m}$ its maximal ideal, $k = A/\mathfrak{m}$ its residue field, the A-module $\mathfrak{m}/\mathfrak{m}^2$ is annihilated by $\mathfrak{m}$ and therefore has the structure of a k-vector space. If $\mathfrak{m}$ is finitely generated, the images in $\mathfrak{m}/\mathfrak{m}^2$ of a set of generators of $\mathfrak{m}$ will span $\mathfrak{m}/\mathfrak{m}^2$ as a vector space and therefore $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is finite (By 2.8).

Proposition 0.3. *Let A be an Artin local ring. Then the following are equivalent:*

1. *Every ideal in A is principal*
2. *The maximal ideal $\mathfrak{m}$ is principal.*
3. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$

Proof. (1) $\implies$ (2), (2) $\implies$ (3) are trivial. We show (3) $\implies$ (1).

If $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 0$, then $\mathfrak{m} = \mathfrak{m}^2$, hence $\mathfrak{m} = 0$ by Nakayama's lemma, and therefore, A is a field, we are good.

If $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$, then $\mathfrak{m}$ is principal by Proposition 2.8. Then let $\mathfrak{m} = (x)$. Let $\mathfrak{a}$ be an ideal of A, not (0) or (1). We have $\mathfrak{m} = \mathfrak{R}$, hence $\mathfrak{m}$ is nilpotent by (8.4) and there exists r with $\mathfrak{a} \subseteq \mathfrak{m}^r$ and $\mathfrak{a} \not\subseteq \mathfrak{m}^{r+1}$. Hence there exists $y \in \mathfrak{a}$ such that $y = ax^r, y \notin (x^{r+1})$. Thus $a \notin (x)$ and a is a unit in A. Hence $x^r \in \mathfrak{a}$, $\mathfrak{m}^r = (x^r) \subseteq \mathfrak{a}$. Thus $\mathfrak{m}^r = (x^r) = \mathfrak{a}$. Hence principal. □

Example 0.4. The rings $\mathbb{Z}/(p^n)$ (p prime), $k[x]/(f^n)$ (f is irreducible) satisfy the conditions of Structure Theorem. However, the Artin local ring $k[x^2, x^3]/(x^4)$ does not, since $\mathfrak{m}$ is generated by x^2 and $x^3 \mod x^4$, so $\mathfrak{m}^2 = 0$ and $\dim(\mathfrak{m}/\mathfrak{m}^2)$.