Modular Forms

Ting Gong

May 18, 2020

1 Dimension of Modular Forms

Definition 1.1. A *modular form of weight k* for $\mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function f on $\mathcal{H}$ satisfying $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and is holomorphic at the cusp ∞.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, $f(z+1) = f(z)$. Then we have a Fourier expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$. Below we denote $q = e^{2\pi i n z}$.

Definition 1.2. A modular form that vanishes at ∞ is called a *cusp form.*

We denote the space of modular forms of weight k for $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ as $M_k(\Gamma(1))$ and the space of cusp forms as $S_k(\Gamma(1))$.

Definition 1.3. An *automorphic function* for Γ is a meromorphic function f on $\mathcal{H}$ and at ∞ such that $f\left(\frac{az+b}{cz+d}\right) = f(z)$

Then f is a meromorphic function on the compact Riemann surface $\Gamma(1)/\mathcal{H}^*$. If f doesn't have a pole, then by Louiville theorem and maximal modulus principle, f is constant.

Notice if $f_1, f_2 \in M_k(\Gamma(1))$, then

$$\frac{f_1}{f_2}\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^k f_1(z)}{(cz+d)^k f_2(z)} = \frac{f_1}{f_2}$$

Therefore, f_1/f_2 is automophic.

Proposition 1.4. *Let X be a compact Riemann surface, $P_1, \ldots, P_n \in X$, let $r_1, \ldots, r_n$ be positive integers. Let V be the vector space of meromorphic functions on X, which are holomorphic besides possibly at P_m, and which are holomorphic or else have poles of order at most r_m at P_m. Then the space V has dimension at most $r_1 + \ldots + r_m + 1$.*

Proof. Let $r = r_1 + \ldots + r_m$, pick a coordinate function $t = t_j$ in a neighborhood of P_j with respect to which P_j is the origin. If $\phi \in V$, it has Laurent expansion, $\phi(t) = a_{j,-r_j} t^{-r_j} + a_{j,-r_j+1} t^{-r_j+1} + \ldots$. We associate ϕ with $v \in \mathbb{C}^r$ whose entries are the Taylor coefficients. If $\phi_1, \ldots, \phi_N \in V$, $N > r$, then $c_1, \ldots, c_N$ are not all zero with $\sum c_j v_j = 0$. Thus $\sum c_j \phi_j$ has no poles. Then since above is meromorphic on a compact Riemann surface, it is constant. Thus any vector subspace of V having dimension greater than r contains a constant function. Thus $\dim V \leq r + 1$. □

Proposition 1.5. *The space $M_k(\Gamma(1))$ is finite dimensional.*

Proof. Let $f_0 \in M_k(\Gamma(1))$ be nonzero. Let X be the compactification of $\Gamma(1)/\mathcal{H}$. Let $P_1, \ldots, P_m$ be zeroes of f_0, let $r_1, \ldots, r_m$ be the orders of zeroes of f_0 at these points. If $f \in M_k(\Gamma(1))$, then by our remark before, f/f_0 is automorphic. Moreover, $f \mapsto f_0$ is an isomorphism of $M_k(\Gamma(1))$ and V in the last proposition. Thus $M_k(\Gamma(1))$ is finite dimensional. □

2 Jacobi's Triple Product Formula

Definition 2.1. Let k be even, $k \geq 4$, the *Eisenstein series* is defined as

$$E_k(z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} (mz + n)^{-k}$$

We notice that the Eisenstein series is absolutely convergent since

$$E_k(z) \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (mz + n)^{-k} dm dn = 4 \int_{0}^{\infty} \int_{0}^{\infty} (mz + n)^{-k} dm dn < \infty$$

since $k \geq 4$, and after integration, the $k - 2 \geq 2$.

Definition 2.2. Let $r \in \mathbb{C}$, the *divisor sum* is defined as

$$\sigma_r(n) = \sum_{d|n} d^r$$

Proposition 2.3. *The Eisenstein series is a modular form.*

Proof. We first show the first condition.

$$\begin{aligned} E_k\left(\frac{az+b}{cz+d}\right) &= \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} (m\left(\frac{az+b}{cz+d}\right) + n)^{-k} \\ &= (cz+d)^k \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} (m(az+b) + n(cz+d))^{-k} \\ &= (cz+d)^k \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} ((am+cn)z + (mb+nd))^{-k} \end{aligned}$$

Since c, d are coprime, $(m, n) \mapsto (ma + nc, mb + nd)$ permutes $\mathbb{Z} \times \mathbb{Z}$. Thus we see $E_k(z)$ satisfies the first condition of a modular form. It suffices to show that it is holomorphic at ∞. To do this, we compute its Fourier expansion. When $m = 0$, $E_k(z) = \zeta(k)$. When $m \neq 0$, since k is even, ± 1 contributes equally. Thus, we only consider $m > 0$.

$$\hat{f}(n) = \int_{-\infty}^{\infty} (mz + n)^{-k} e^{2\pi inz}$$

Then by the residue theorem,

$$\hat{f}(n) = 2\pi i \text{res}(e^{2\pi inz}(mz - n)^{-k}) = \frac{2\pi i}{(k-1)!} n^{k-1} e^{2\pi imnz}$$

Then by Poisson Summation Formula,

$$E_k(z) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi imnz} = \zeta(k) + \frac{(2\pi)^k (-1)^{\frac{n}{2}}}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $q = e^{2\pi iz}$. Therefore, we see that it is holomorphic at ∞. $\square$

For a given k, either $S_k(\Gamma(1)) = M_k(\Gamma(1))$ or $\dim S_k(\Gamma(1)) + 1 = \dim M_k(\Gamma(1))$, since if these is a modular form of weight k, either the constant is zero, or we can substract by a multiple. For $k \geq 4$, we see that there is an Eisenstein series with nonzero constant term. Therefore, $\dim M_k(\Gamma(1)) = \dim S_k(\Gamma(1)) + 1$.

We observe that the modular forms form a graded ring. It is easy to show that if $f \in M_k(\Gamma(1))$ and $g \in M_l(\Gamma(1))$, then $fg \in M_{k+l}(\Gamma(1))$.

Example 2.4. We construct example below: let

$$G_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \text{ and } G_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

Clearly, G_4 has weight 4 and G_6 has weight 6. Then we define

$$\Delta(z) = \frac{1}{1728}(G_4^3 - G_6^2) = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

which becomes a cusp form of weight 12.

Theorem 2.5 (Jacobi's Triple Formula)**.**

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}x)(1 + q^{2n-1}x^{-1})$$

Proof. Let

$$\nu(z, w) = \sum_{n=-\infty}^{\infty} q^{n^2} x^n, q = e^{2\pi iz}, x = e^{2\pi iw}$$

$$\nu(z,w+2z)=\sum_{n=-\infty}^{\infty}q^{n^2}(xq^2)^n=\sum_{n=-\infty}^{\infty}q^{n^2+2n}x^n=(qx)^{-1}\sum_{n=-\infty}^{\infty}q^{(n+1)^2}x^{n+1}=(qx)^{-1}\nu(z,w)$$

And we let

$$P(z,w)=\prod_{n=1}^{\infty}(1+q^{2n-1}x)(1+q^{2n-1}x^{-1})$$

$$P(z,w+2z)=\prod_{n=1}^{\infty}(1+q^{2n+1}x)(1+q^{2n-3}x^{-1})=(qx)^{-1}P(z,w)$$

Therefore, let $\Lambda\subset\mathbb{C}$ be the lattice $\{2mz+n|m,n\in\mathbb{Z}\}$ and $f(w)=\frac{\nu(z,w)}{P(z,w)}$, then $f(z)$ is an elliptic function over Λ. Assume $P(z,w)=0$, for fixed z. Then some factor of P is zero. Namely, $q^{2n-1}x=0$ or $q^{2n-1}x^{-1}=0$, for some n. Then $2\pi iz(2n-1)\pm 2\pi iw=k\pi i$, where k is odd. Therefore, $w=\pm z+\lambda+\frac{1}{2}$, where $\lambda\in\Lambda$. Thus these w are zeroes of $P(z,w)$.

We show that these w are also zeroes of $\nu(z,w)$. Since $n^2(2\pi iz)+n(2\pi iw)\mod 2=\pi i(2zn^2+\pm 2nz+2n\lambda+n)\mod 2=n\pi i\mod 2$. Thus it is a series permuting between -1 and 1. And the sum gives 0. Therefore, we see that $f(w)$ doesn't have a pole. Hence, $f(w)$ is a constant, say $\phi(q)$. Thus $\nu(z,w)=\phi(q)P(z,w)$.

Next, it suffices to show $\phi(q)=\prod_{n=1}^{\infty}(1-q^{2n})$. Consider

$$\nu(4z,\frac{1}{2})=\sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2}=\sum_{n=-\infty}^{\infty}e^{\frac{n\pi i}{2}}q^{n^2}=\nu(z,\frac{1}{4})$$

Thus, we divide the two equations and we have $\phi(q)=\frac{P(4z,\frac{1}{2})}{P(z,\frac{1}{4})}\phi(q^4)$, now we compute

$$\frac{P(4z,\frac{1}{2})}{P(z,\frac{1}{4})}=\prod_{n=1}^{\infty}(1-q^{4n-2})(1-q^{8n-4})$$

Then as $q\to 0$ $\phi(q)\to 1$. Therefore, $\phi(q)=\prod_{n=1}^{\infty}(1-q^{2n})$. □

3 Dimension of Cusp Forms

We use Jacobi's triple product formula, replacing q with $q^{\frac{3}{2}}$ and x with $-q^{\frac{-1}{2}}$. Then

$$\sum_{n=-\infty}(-1)^n q^{\frac{(6n+1)^2}{24}}=q^{\frac{1}{24}}\sum_{n=-\infty}(-1)^n q^{\frac{3n^2+n}{2}}=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^{3n})1-q^{3n-1})(1-q^{3n-2})=\prod_{n=1}^{\infty}(1-q^n)$$

Definition 3.1. The *Dedekind eta function* is defined as

$$\eta(z)=q^{\frac{1}{24}}\prod(1-q^n)=\sum_{-\infty}^{\infty}\chi(n)q^{\frac{n^2}{24}}$$

Where $\chi(n)=1$ if $n\equiv\pm 1\mod 12$, -1 if $n\equiv\pm 5\mod 12$ and 0 otherwise.

Proposition 3.2. *If $\gamma \in \Gamma(1)$ then there exists a 24th root of unity $\epsilon(\gamma)$ such that $\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(\gamma)(cz+d)^{\frac{1}{2}}\eta(z)$.*

Proof. Since $\Gamma(1)$ is spanned by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, it suffices to check these two. $T(q^{\frac{1}{24}}) = e^{\frac{2\pi i}{24}} q^{\frac{1}{24}}$. Then we are done. Since $\eta(z) = \theta_\chi(\frac{-iz}{12})$, we have $\tau(\chi) = 2\sqrt{3}$ and $N = 12$. Thus $\sqrt{-iz}\eta(z) = \eta(-\frac{1}{z})$. Thus, we are done. $\square$

Therefore, it is reasonable to get rid of the root of unity by rising to 24th power. Let $\Delta(z) = \eta(z)^2 4 = q\prod_{n=1}^{\infty}(1-q^n)^{24}$, then $\Delta(z)$ is a cusp form of weight 12. Since it is defined by a convergent infinite product, $\Delta(z) \neq 0$.

Proposition 3.3. *The space $S_k(\Gamma(1))$ is one dimensional, spanned by Δ, where $\Delta = \frac{1}{1728}(G_4^3 - G_6^2)$.*

Proof. Let $f \in S_k(\Gamma(1))$. Then f/Δ is an automorphic function, having no poles in $\mathcal{H}$. It is also holomorphic at the cusp since f vanishes. Therefore, f/Δ has no pole, thus a constant. Thus $S_k(\Gamma(1))$ is generated by Δ. Thus $\frac{1}{1728}(G_4^3 - G_6^2) = c\Delta$. Thus by comparing the Fourier coefficients, $c = 1$. $\square$

Proposition 3.4. *Suppose k is an even positive integer. $k = 12j + r$, where $0 \leq r \leq 10$. Then $\dim M_{12j+r}(\Gamma(1)) = j + 1$ if $r = 0, 4, 6, 8, 10$. Otherwise it is j. And the ring $\bigoplus_{k=0}^{\infty} M_k(\Gamma(1))$ is generated by G_4 and G_6.*

Proof. We do induction over j. Let $j = 0$, we check when $k = 4, 6, 8, 10$, $M_k(\Gamma(1))$ is one dimensional. Let $h = 6(12-k)$. If $f \in M_k(\Gamma(1))$ is not in the one dimensional space spanned by E_k, we can substract the constant Fourier coefficient and assume f is in $S_k(\Gamma(1))$. Consider $E_h(f/\Delta)^6$. Then we know that E_h has weight h and $(f/\Delta)^6$ has weight $6(k-12) = -h$. Thus this is an automorphic form with no poles. Thus this is constant. Hence $E_h = c\Delta^6/f^6$. Thus E_h has no zeroes on $\mathcal{H}$. Now let $h = 12H$, where $H = 1, 2, 3, 4$. Then Δ^H/E_h is an automorphic function with no poles but a zero of order H at ∞, but it contradicts the definition of a cusp form. This it must be one dimensional spanned by E_k.

Then we show $M_2(\Gamma(1))$ has dimensional 0. If $f \in M_k(\Gamma(1))$, then $fE_4 \in M_6(\Gamma(1))$. So $fE_4 = cE_6$. for some c. Let $\rho = e^{2\pi i/3}$, if $3 \nmid k$, $f \in M_k(\Gamma(1))$, then let $\gamma = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\gamma(\rho) = \rho$. Then $f(\gamma(\rho)) = f(\rho) = (-z)^k f(\rho)$. Thus $f(\rho) = 0$. Thus $E_4(\rho) = 0$. Thus $E_6(\rho) = 0$, thus $\Delta(\rho) = 0$, contradiction. Thus $M_2(\Gamma(1)) = 0$. $M_0(\Gamma(1))$ is one dimensional is clear, consisting of constants.

Let $j \geq 1$, multiplying Δ is an isomorphism between $M_{k-12}(\Gamma(1))$ and $S_k(\Gamma(1))$. Injection is clear. Let $f \in S_k(\Gamma(1))$, then f/Δ have no poles. Thus is in $M_{k-12}(\Gamma(1))$. Thus we use induction step and our former discussion about the dimensions between $M_k(\Gamma(1))$ and $S_k(\Gamma(1))$ to show this.

Then since σ_2 and σ_3 are clearly algebraically independent, we know G_4 and G_6 are algebraically independent as $G_k(z) = \zeta^{-1}(k)E_k(z)$.

Then let R be the subring generated by G_4 and G_6. Since M_8, M_{10} are one dimensional, they are clearly generated by E_4^2 and E_4E_6. Thus $M_k \subset R$ for $k \leq 10$. Since $\Delta \in R$, let k be the first even integer such that $M_k \not\subset R$, $k \geq 12$. Since $S_k = \Delta M_{k-12} \subset R$, and $E_4^s E_6^k \in R$, with $4k + 6s = k$, then we know R contains M_k. □

4 Petersson Inner Product and L-function

We define an inner product on $S_k(\Gamma(1))$:

Definition 4.1. Let $f, g \in S_k(\Gamma(1))$, the *Petersson Inner Product* is defined as

$$\langle f, g\rangle = \iint\limits_{\Gamma(1)/\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

Clearly, since $\Im(\gamma(z)) = \frac{\Im(z)}{|cz+d|^2}$, where $\gamma \in \Gamma(1)$, we know the inner product is invariant under our action by γ.

If $n > 0$, $q^n \to 0$ as $z \to \infty$. Since a cusp form has a Fourier expansion $\sum a_n q^n$ with $a_n \neq 0$ for $n > 0$. A cusp form decays quickly as $y \to \infty$. Thus the above definition is well defined.

Lemma 4.2. *If at least one of f, g is a cusp form, the Petersson inner product is well-defined.*

Proof. Since we can integrate over a compact set containing each cusp, it suffices to prove the lemma for the cusp at infinity. Since $f(z)\overline{g(z)} \in O(e^{-cy})$, with one of them a cusp form, the integral is dominated by $\int e^{-cy}y^{k-2} < \infty$. □

Definition 4.3. Let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be a modular form. We define $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$ be the *L-function* of f.

Proposition 4.4. *If f is cuspidal, its Fourier coefficients satisfy $a_n \leq Cn^{k/2}$ for some C independent of n. (This is not necessarily true for f not being a cusp form, but the L-function should be convergent for larger $\Re(s)$)*

Proof. It is clear that $|f(z)y^{k/2}|$ is invariant under our action. Since f is a cusp form, the function decays as z approaches the cusp. Then it is bounded on the fundamental domain. Thus $\exists C_1$ such that $|f(z)y^{k/2}| \leq C_1$. Fix y, we have

$$|a_n|e^{-2\pi ny} = |\int_0^1 f(x+iy)e^{-2\pi in(x+iy)}dx|e^{-2\pi ny} \leq \int_0^1 |f(x+iy)e^{-2\pi inx}|dx \leq C_1 y^{\frac{-k}{2}}$$

Then pick $y = \frac{1}{n}$ to get $a_n < e^{2\pi}C_1 n^{k/2}$, which proves our theorem. □

Proposition 4.5. *Let $\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f)$, then $\Lambda(s, f)$ extends to an analytic function of s if f is a cusp form. Otherwise it has simple poles at $s = 0$ and $s = k$, where $\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f)$.*

Proof. If f is a cusp form, $f(iy) \to 0$ as $y \to \infty$. When $\gamma = S$, $f(iy) = (-1)^{k/2}y^{-k}f(i/y)$. Then $f(iy) \to 0$ when $y \to 0$. Thus $\int_0^{\infty} f(iy)y^s\frac{dy}{y}$ is convergent for all s. Thus we see that this is analytic. If $\Re(s)$ is large, $\int_0^{\infty} e^{-2\pi ny}y^s\frac{dy}{y} = (2\pi)^{-s}\Gamma(s)$, we know that the above function is indeed $\Lambda(s, f)$. Then we use action by S and replace $\frac{1}{y}$ by y, we have the recursive definition. Moreover, since 0 gives a pole for f not cuspital, it is clear that it has another pole at $s = k$. □