Rankin-Selberg Method

Ting Gong

June 7, 2020

1 Eisenstein Series

Definition 1.1. The *Eisenstein series* for $\mathrm{SL}(2,\mathbb{Z})$ is defined as

$$E(z,s)=\frac{1}{2}\pi^{-s}\Gamma(s)\sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)}\frac{y^s}{|mz+n|^{2s}}$$

By basic complex analysis, when $\Re(s)>1$, the Eisenstein series converges absolutely. Moreover, by direct computation, we see that it is strictly automorphic.

Definition 1.2. The *K-Bessel function* is defined as

$$K_s(y)=\frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\frac{dt}{t}$$

Remark 1.3. We develop some properties of the K-Bessel function: If $y>0$, as $t\to 0$ or ∞, $K_s(y)\to 0$. Thus it is convergent for all s.

Let $a=y/2$, $b=t+t^{-1}$, if $a,b>2, ab>a+b$. Thus $e^{-ab}<e^{-a}e^{-b}$.

$$K_s(y)=\frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\frac{dt}{t}\leq\frac{1}{2}\int_0^\infty e^{-y/2}e^{-(t+t^{-1})}t^s\frac{dt}{t}=e^{-y/2}K_{\Re(s)}(2)$$

Moreover, it is easy to see that $K_s(y)$ is invariant under $t\mapsto t^{-1}, s\mapsto -s$. We have $K_s(y)=K_{-s}(y)$.

We then compute:

$$\begin{aligned}\left(\frac{y}{\pi}\right)^s\Gamma(s)\int_{-\infty}^\infty(x^2+y^2)^{-s}e^{2\pi irx}dx&=\int_0^\infty t^se^{-t}\frac{dt}{t}\int_{-\infty}^\infty\left(\frac{y}{\pi(x^2+y^2)}\right)^s e^{2\pi irx}dx\\
=\int_{-\infty}^\infty\int_0^\infty e^{-t}\left(\frac{yt}{\pi(x^2+y^2)}\right)^s e^{2\pi irx}\frac{dt}{t}dx&=\int_0^\infty\int_{-\infty}^\infty e^{-\pi t(x^2+y^2)/y}t^se^{2\pi irx}dx\frac{dt}{t}\\
=\int_0^\infty\sqrt{\frac{y}{t}}e^{-y\pi r^2/t}t^s\frac{dt}{t}&=2|r|^{s-1/2}\sqrt{y}K_{s-1/2}(2\pi|r|y)\end{aligned}$$

Theorem 1.4. *$E(z,s)$ has meromorphic continuation to all s. It is analytic except at $s=1$ and $s=0$, where it has simple poles with residue $1/2$ at $s=1$. Moreover, $E(z,s)=E(z,1-s)$, and $E(x+iy,s)=O(y^\sigma)$ as $y\to\infty$, where $\sigma=\max(\Re(s),1-\Re(s))$.*

Proof. Since $E(z,s)$ is automorphic, $E(z,s) = E(z+1,s)$. Then, we compute its Fourier expansion. Let $E(z,s) = \sum_{-\infty}^{\infty} a_r(y,s)e^{2\pi irx}$. We compute its Fourier coefficients. $a_r(y,s) = \int_0^1 E(x+iy,s)e^{-2\pi irx}dx$. When $m = 0$, this term contributes to a_0. Since n and $-n$ contributes equally, we have contribution

$$\pi^{-s}\Gamma(s)y^s \sum_{n=1}^{\infty} n^{-2s} = \pi^{-s}\Gamma(s)\zeta(2s)y^s$$

When $m \neq 0$, since (m,n) and $(-m,-n)$ contributes equally, the contribution is

$$\begin{aligned}
&\pi^{-s}\Gamma(s)y^s \sum_{n=1}^{\infty}\sum_{-\infty}^{\infty}\int_0^1 [(mx+n)^2+m^2y^2]^{-s}e^{2\pi irx}dx \\
&= \pi^{-s}\Gamma(s)y^s \sum_{n=1}^{\infty}\sum_{n \mod m}\int_{-\infty}^{\infty} [(mx+n)^2+m^2y^2]^{-s}e^{2\pi irx}dx \\
&= \pi^{-s}\Gamma(s)y^s \sum_{n=1}^{\infty} m^{-2s}\sum_{n \mod m} e^{2\pi irn/m}\int_{-\infty}^{\infty}(x^2+y^2)^{-s}e^{2\pi irx}dx \\
&= \pi^{-s}\Gamma(s)y^s \sum_{m|r} m^{1-2s}\int_{-\infty}^{\infty}(x^2+y^2)^{-s}e^{2\pi irx}dx
\end{aligned}$$

Therefore, by our remark, one can compute a_r easily as a sum of the above two expressions.

$$a_0 = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}$$

$$a_r = 2|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y)$$

Therefore, each term has analytic continuation to all s besides a_0 has poles at $s = 0$ and $s = 1$. Since the K-Bessel function decays, this converges. And the functional equation can be observed by $a_n(y,s) = a_n(y,1-s)$. Thus the statement is proven. □

Since summing (m,n) is equivalent to sum (Nc,Nd) over N being a positive integer and (c,d) over all coprime numbers. We associate a coset in $\Gamma_\infty\backslash\Gamma(1)$ by (c,d) being the bottom row. Then $\frac{y^s}{|mz+n|^{2s}} = N^{-2s}\Im(\gamma(z))^s$. Then

$$E(z,s) = \pi^{-s}\Gamma(s)\sum_{\gamma\in\overline{\Gamma_\infty}\backslash \mathrm{PSL}(2,\mathbb{Z})}\Im(\gamma(z))^s$$

2 Rankin-Selberg Method

Let ϕ be automorphic on $\mathcal{H}$. Let $\phi(x+iy) = O(y^{-N})$ for all $N > 0$ as $y \to \infty$. Since $\phi(z+1) = \phi(z)$, we have Fourier expansion $\phi(z) = \sum_{-\infty}^{\infty}\phi_n(y)e^{2\pi inx}$, $\phi_n(y) = \int_0^1 \phi(x+iy)e^{-2\pi inx}dx$. Let the Mellin transform of ϕ_0 be $M(s,\phi_0) = \int_0^\infty \phi_0(y)y^s\frac{dy}{y}$. Then since ϕ is bounded on the fundamental domain, ϕ_0 is bounded and decays as $y \to \infty$, thus the Mellin transform is absolute convergent when $\Re(s) > 0$. Let

$$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)M(s-1,\phi_0)$$

Proposition 2.1.

$$\Lambda(s) = \int_{\Gamma(1)\backslash\mathcal{H}} E(Z,s)\phi(z)\frac{dxdy}{y^2}$$

Then Λ has meromorphic continuation to all s with at most simple poles at $s=1, s=0$, and

$$res(\Lambda(s))|_{s=1} = \frac{1}{2}\int_{\Gamma(1)\backslash\mathcal{H}} \phi(z)\frac{dxdy}{y^2}$$

Proof. If we prove the first identity, then by Theorem 1.4, we have the rest. Let $\Re(s) > 1$, consider

$$\begin{aligned}
&\pi^{-s}\Gamma(s)\zeta(2s) \sum_{\gamma\in\overline{\Gamma_\infty}\backslash \mathrm{PSL}(2,\mathbb{Z})} \int_{\Gamma(1)\backslash\mathcal{H}} \Im(\gamma(z))^s\phi(\gamma(z))\frac{dxdy}{y^2} \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_{\Gamma_\infty\backslash\mathcal{H}} \Im(z)^s\phi(z)\frac{dxdy}{y^2} \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_0^\infty\int_0^1 y^s\phi(x+iy)y^{-1}\frac{dxdy}{y}
\end{aligned}$$

which is what we desired. ∎

Remark 2.2. Let $f(z) = \sum A(n)q^n, g(z) = \sum B(n)q^n$ are modular forms. Let $\phi(z) = f(z)\overline{g(z)}y^k$. Then $\phi_0(y) = \sum_{n=0}^\infty A(n)\overline{B(n)}e^{-4\pi ny}y^k$ by direct computation. Then

$$M(s,\phi_0) = (4\pi)^{-(s+k)}\Gamma(s+k)\sum_{n=0}^\infty A(n)\overline{B(n)}n^{-(s+k)}$$

Then since $B(n)$ is self-adjoint:

$$\Lambda(s) = 4^{-s-k+1}\pi^{-2s-k+1}\Gamma(s)\Gamma(s-k+1)\zeta(2s)\sum_{n=1}^\infty A(n)B(n)n^{-s-k+1}$$

Then we let $L(s, f\times g) = \zeta(2s-2k+2)\sum_{n=1}^\infty A(n)B(n)n^{-s}$, then

$$\Lambda(s,f\times g) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)L(s,f\times g) = \pi^{1-k}\Lambda(s-k+1) = \Lambda(2k-1-s, f\times g)$$

Then we observe that this has simple poles at $s=k, s=k-1$, with resudue at $s=k$ being $\frac{1}{2}\pi^{1-k}\langle f,g\rangle$.

Then the Rankin-Selberg Method can help us prove the following Lemma and Theorem (mostly by computation, so we know the Euler product of $L(s, f\times g)$)

Lemma 2.3. *If $\sum_{r=0}^\infty A(r)x^r = (1-\alpha_1 x)^{-1}(1-\alpha_2 x)^{-1}$ and $\sum_{r=0}^\infty B(r)x^r = (1-\beta_1 x)^{-1}(1-\beta_2 x)^{-1}$, then*

$$\sum_{r=0}^\infty A(r)B(r)x^r = (1-\alpha_1\beta_1\alpha_2\beta_2 x^2)^{-1}\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_j x)^{-1}$$

Theorem 2.4.

$$L(s,f\times g) = \prod_p\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i(p)\beta_j(p)p^{-s})^{-1}$$