

Math 20580
Practice Midterm 2
March 5, 2015

Name: Solutions
Instructor: _____
Section: _____

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an \times through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1. a b c d e
 2. a b c d e
 3. a b c d e
 4. a b c d e
 5. a b c d e
 6. a b c d e
 7. a b c d e
 8. a b c d e
-

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Let V be the vector space of all functions $f(x)$ where $f: \mathbb{R} \rightarrow \mathbb{R}$.

Which of the following are subspaces of V ?

- A. the constant functions;
- B. functions with $\lim_{x \rightarrow \infty} f(x) = 3$;
- C. functions with $f(1) = 1$;
- D. functions with $f(0) = 0$.

- (a) A, B, C and D (b) A, B and C only (c) B, C and D only
(d) B and D only (e) A and D only.

B & C are not subspaces.

*For example, if $\lim_{x \rightarrow \infty} f(x) = 3$ then
 $\lim_{x \rightarrow \infty} 2f(x) = 6$.*

2. Let \mathbb{P}_2 be the vector space of polynomials of degree at most 2, and let \mathcal{B} be the basis

$$\mathcal{B} = \{1, 1+x, 1+x^2\}.$$

Find the \mathcal{B} -coordinates $[p]_{\mathcal{B}}$ of the polynomial $p(x) = (1-x)^2$.

- (a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\begin{aligned} p(x) &= 1 - 2x + x^2 \\ &= 2(1) - 2(1+x) + (1+x^2) \end{aligned}$$

3. Let H be a subspace of a vector space V , and suppose that V has dimension d . Which of the following statements are true?

- A. $\dim(H) \leq \dim(V)$; ✓
- B. a linearly independent set of vectors in H is also linearly independent in V ; ✓
- C. d vectors which span V will be linearly independent; ✓
- D. d vectors which span H will also span V . ✗

- (a) A, B, C and D (b) A, B and C only (c) B, C and D only
(d) B and D only (e) A and D only.

Only D is false. Spanning H does not mean vectors span all of V .

4. A linear transformation

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

has outputs $T \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Let $B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$ be a basis of \mathbb{R}^2

$$\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_B = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix}.$$

$$\text{Therefore } T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = T \left(\frac{2}{5} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right) = \frac{2}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

5. The vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for the matrix

$$\begin{bmatrix} 2 & 5 & 1 \\ 1 & 7 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

What is the corresponding eigenvalue?

- (a) 3 (b) 1 (c) 0 (d) -1 (e) 2

$$\begin{pmatrix} 2 & 5 & 1 \\ 1 & 7 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

so the eigenvalue is 3.

6. What are the eigenvalues of the matrix $\begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$?

- (a) 0, 1 (b) 5, -1 (c) 1, 3 (d) 0, 2 (e) -1, -3

$$\begin{aligned} \det(A - \lambda I) &= (5 - \lambda)(-1 - \lambda) + 8 \\ &= -5 - 4\lambda + \lambda^2 + 8 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 3)(\lambda - 1) \end{aligned}$$

so $\lambda = 1, 3$

7. Suppose A is a 3×3 matrix, that has $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as an eigenvector with eigenvalue 2, and

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as an eigenvector with eigenvalue -1 . Compute $A^3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

- (a) $\begin{bmatrix} 8 \\ 16 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 16 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
 (e) cannot be computed from the given information

$$\begin{aligned} A^3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} &= A^3 \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = A^3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + A^3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 8 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ -1 \end{pmatrix} \end{aligned}$$

8. Let P_3 be the vector space of polynomials of degree at most 3. Find the dimension of the subspace of P_3 spanned by $1 + x^2$, $x + 2x^2 + x^3$ and $1 + x + x^3$.

- (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

The coordinates of the polynomials relative to the basis $B = \{1, x, x^2, x^3\}$ are

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ respectively.

The matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

There are pivots in each column so the polynomials are linearly independent.

Part II: Partial credit questions (11 points each). Show your work.

9. Find a basis for the Row space, $\text{Row}(A)$, of the matrix

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 2 & 3 \\ 1 & 1 & -5 & 6 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$$

$$A \rightsquigarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -7 & 3 \\ 0 & -1 & -3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -10 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The non-zero rows form a basis, namely

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -10 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

10. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ be two bases of \mathbb{R}^2 .

Find the change of coordinate matrix $\mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ sending a \mathcal{B} coordinate vector $[\vec{x}]_{\mathcal{B}}$ to the \mathcal{C} coordinate vector $[\vec{x}]_{\mathcal{C}}$.

$$\text{Let } B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{aligned} \text{Then } \mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}} &= C^{-1}B = \frac{1}{(-1)} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= - \begin{pmatrix} 2 & 4 \\ 5 & 11 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -4 \\ -5 & -11 \end{pmatrix} \end{aligned}$$

11. Is the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$ diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If A is not diagonalizable, explain why not.

(Hint: The eigenvalues of A are 1 and 3.)

Eigenspace for $\lambda = 1$ are solns. to $(A - I)x = 0$

$$A - I = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ \& basis is } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Eigenspace for $\lambda = 3$ are solns. to $(A - 3I)x = 0$

$$A - 3I = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & 0 \\ -1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ \& basis is } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Take } P = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ \& } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

12. Let P_3 be the vector space of polynomials of degree at most 3 and P_2 be the space of polynomials of degree at most 2.

Consider the linear transformation

$$T : P_3 \rightarrow P_2$$

$$\text{given by } T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

- (a) Write down bases \mathcal{B}_3 and \mathcal{B}_2 of P_3 and P_2 respectively.

$$\mathcal{B}_3 = \{1, x, x^2, x^3\}$$

$$\mathcal{B}_2 = \{1, x, x^2\}$$

- (b) Find the matrix of T relative to the bases \mathcal{B}_3 and \mathcal{B}_2 .

$$\left. \begin{array}{l} T(1) = 0 \\ T(x) = 1 \\ T(x^2) = 2x \\ T(x^3) = 3x^2 \end{array} \right\} \text{The matrix is } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$