Name: **Version #1**

Instructor: Bullwinkle

Exam II

October 31, 2019

- The Honor Code *is* in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 8 pages *of* the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
11.	
12.	
13.	
Total	

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3.	(a)	(b)	(c)	(d)	(e)
4	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
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Please do NOT write in this box.

Multiple Choice ______________

11. ______________

12. ______________

13. ______________

Total ______________

1.(6pts) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

(a) $0, 2, 4$ (b) $0, 0, 2$ (c) $0, 1, 2$ (d) $2, 2, 2$ (e) $2, 4, 4$

Solution. Computing the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ yields via cofactor expansion along the first column $(2-\lambda)^3 - 4(2-\lambda) = (2-\lambda)((2-\lambda)^2 - 4) = (2-\lambda)(\lambda)(\lambda-4)$. Thus, the eigenvalues are $0, 2, 4$.

2.(6pts) Let A, B, C be 4×4 matrices with $\det(A) = \frac{1}{5}$, $\det(B) = 5$, and $\det(C) = 8$. What is $\det(2A^T B^2 C^{-1})$?

(a) 10 (b) $\frac{5}{4}$ (c) $\frac{1}{4}$ (d) 124 (e) 620

Solution. We compute $\det(2A^T B^2 C^{-1}) = 2^4 \det(A) \det(B)^2 \det(C)^{-1} = 2^4 \cdot \frac{1}{5} \cdot 5^2 \cdot \frac{1}{8} = 10$

3.

3.(6pts) Find the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ between the following bases of $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

(a) $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$

Solution. We row reduce the matrix $\begin{bmatrix} 1 & 2 & 6 & 2 \\ 2 & 3 & 9 & 4 \end{bmatrix}$ to find the change-of-coordinates matrix to be $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$.

4.(6pts) The vector $\vec{v} = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix}$ is a complex eigenvector of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$. What is the corresponding complex eigenvalue?

(a) $4+3i$ (b) $3+2i$ (c) $3-4i$ (d) $4-3i$ (e) $3-2i$

Solution. By assumption, $A\vec{x} = \lambda\vec{x}$, and we want to find λ. We compute $A\vec{x} = \begin{bmatrix} -13+9i \\ 8+6i \end{bmatrix}$. Examining the second entry, we thus see that $\lambda = 4+3i$.

4.

5.(6pts) Which subsets of $\mathbb{P}_3$ are linearly independent?

I. $\{1+t, 1-t, t-t^3, (1-t)^2, 1+2t-t^2+5t^3\}$

II. $\{t^3, (t-1)^2\}$

III. $\{(t-1)^3, (t-2)^3, (t-3)^3, (t-4)^3, (t-5)^3\}$

IV. $\{1+t+3t^2, 1, 1+2t\}$

(a) II and IV

(b) II, III, and IV

(c) all are linearly independent

(d) I, II, and III

(e) none are linearly independent

Solution. Both (I) and (III) have 5 vectors, so they are linearly dependent sets. On the other hand, we claim that (II) and (IV) are linearly independent. Indeed, t^3 is not a scalar multiple of $(t-1)^2$, which shows this for (II), and choosing standard coordinates $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$ we see that the matrix formed from the column vectors of (IV) is upper triangular with all nonzero diagonal entries, hence invertible.

6.(6pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2, T(x) = Ax$ with $A = \begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}$. Which of the following bases $\mathcal{B}$ for $\mathbf{R}^2$ gives a diagonal $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ for T?

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$

Solution. A basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal is given by the eigenvectors for A. We compute the eigenvalues to be 3 and 5, and then the corresponding eigenvectors to be $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

7.(6pts) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 11 & 0 & 2 \\ 0 & -3 & 0 & 0 \\ 4 & -9 & 6 & 12 \\ 2 & -20 & 0 & 3 \end{bmatrix}$$

(a) 18 (b) 0 (c) 25 (d) -25 (e) -18

Solution. Cofactor expansion along the second row shows the determinant to equal -3 times the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 4 & 6 & 12 \\ 2 & 0 & 3 \end{bmatrix}$. Cofactor expansion along the second column then shows that determinant to equal 6 times $3 - 4 = -1$. We conclude the determinant to equal 18.

8.(6pts) Which of the following sets are subspaces of the indicated vector spaces?

I. The set of all vectors of the form $\begin{bmatrix} s+t \\ 2t \\ 3s \end{bmatrix}$ in $\mathbb{R}^3$

II. The set of all degree 2 polynomials $p(t)$ (i.e., those of the form $at^2 + bt + c$ with $a \neq 0$), inside the vector space $\mathbb{P}_2$ of polynomials with degree ≤ 2.

III. The set of all invertible 2×2 matrices A, inside the vector space of 2×2 matrices.

IV. The set of all 2×2 matrices A of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, inside the vector space $M_{2\times 2}$ of 2×2 matrices.

(a) I and IV are subspaces (b) III and IV are subspaces (c) I, II, and III are subspaces

(d) All are subspaces (e) II and IV are subspaces

Solution. (I) is given by the span of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, so is a subspace. (II) is not a subspace since it doesn't contain the zero vector, which is the zero polynomial. (III) is not a subspace since it doesn't contain the zero vector, which is the 2×2 matrix with all zeroes as entries. (IV) is a subspace as one may check it is closed under vector addition and scalar multiplication.

9.(6pts) Let $\mathbb{P}_2$ be the vector space of degree ≤ 2 polynomials. Which of the following linear transformations $T : \mathbb{P}_2 \to \mathbb{R}^3$ are both one-to-one and onto? Select the answer that corresponds to *all* valid choices.

I. $T(p(t)) = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix}$.

II. $T(p(t)) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}$.

III. $T(p(t)) = \begin{bmatrix} p'(1) \\ p'(2) \\ p'(3) \end{bmatrix}$.

IV. $T(p(t)) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix}$.

(a) I, II, and IV only
(b) None
(c) I, II, and III only
(d) All
(e) III and IV only

Solution. We see that (III) is not one-to-one since $T(1)$ is the zero vector. This rules out all options except for 'None' and 'I, II, and IV only'. On the other hand, (IV) is one-to-one and onto, since if we use the standard basis $\{1, t, t^2\}$ of $\mathbb{P}_2$ to write the linear transformation T as a matrix A, we get $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, which is invertible. Thus, the answer must be 'I, II, and IV only'. We could also check (I) and (II) work via the same method.

10.(6pts) Let A be an $n \times n$ matrix, $n > 1$. Which of the following statements is *false*?

(a) If A is diagonalizable, then A has n distinct eigenvalues.

(b) If A has n distinct eigenvalues, then A is diagonalizable.

(c) If 0 is an eigenvalue of A, then A is not invertible.

(d) If λ is an eigenvalue of A, then $\text{Nul}(A - \lambda I)$ has dimension not equal to 0.

(e) If A is diagonalizable, then $\mathbb{R}^n$ has a basis comprised of eigenvectors for A.

Solution. The identity matrix is diagonalizable, but its eigenvalues are all equal to one, hence the statement 'If A is diagonalizable, then A has n distinct eigenvalues.' is false.

11.(14pts) The eigenvalues of the matrix $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -1 & -1 & 0 \end{bmatrix}$ are 2 and -2.

(i) Find a basis for the eigenspace of A corresponding to the eigenvalue 2.
(ii) Find a basis for the eigenspace of A corresponding to the eigenvalue -2.
(iii) Give a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, or if none exists, explain why. (Note: you do not need to compute P^{-1}.)

Solution. For the 2-eigenspace, we have a basis given by the vectors $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

For the -2-eigenspace, we have a basis given by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus, for (iii) one possible option is given by $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

12.(14pts) Let $\mathbb{P}_2$ be the vector space of degree ≤ 2 polynomials, and consider the two bases

$$\mathcal{B} = \left\{1, t, -2 + t^2\right\}, \quad \mathcal{C} = \left\{1 - 2t + t^2, t^2, 1 + 2t + t^2\right\}.$$

(i) Let $p(t) = 2 - t + t^2$. Find the $\mathcal{B}$-coordinate vector $[p(t)]_{\mathcal{B}}$ of $p(t)$.
(ii) Find the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$, that transforms $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.
(iii) Find the $\mathcal{C}$-coordinate vector $[p(t)]_{\mathcal{C}}$ of $p(t) = 2 - t + t^2$.

Solution. We may express both bases relative to the standard basis $\{1, t, t^2\}$ of $\mathbb{P}_2$ in order to compute. We thus row reduce the 3×6-matrix $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & -1 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 \end{bmatrix}$ to get

$\begin{bmatrix} 1 & 0 & 0 & 1/2 & -1/4 & -1/2 \\ 0 & 1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1/2 & 1/4 & -1/2 \end{bmatrix}$, so $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1/2 & -1/4 & -1/2 \\ -1 & 0 & 3 \\ 1/2 & 1/4 & -1/2 \end{bmatrix}$.

For (i) solve: $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ and for

(iii) solve $\begin{bmatrix} 1 & 0 & 1 & 2 \\ -2 & 0 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ or multiply $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ with the vector obtained in (ii).

13.(12pts) Let $A = \begin{bmatrix} 15 & 5 & 0 & 15 \\ 6 & 2 & 0 & 4 \\ 3 & 2 & 1 & 4 \\ 9 & 6 & 0 & 3 \end{bmatrix}$

(i) Compute the determinant of A.

(ii) Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a vector in $\mathbb{R}^4$ such that $Ax = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 6 \end{bmatrix}$.

Use *Cramer's rule* to find x_1.

Solution. For (i), cofactor expansion along the third column shows that $\det(A)$ equals the determinant of the 3×3 matrix $\begin{bmatrix} 15 & 5 & 15 \\ 6 & 2 & 4 \\ 9 & 6 & 3 \end{bmatrix}$. Factoring out 5 from the first row, 2 from the second row, and 3 from the third row shows that the determinant is then 30 times the determinant of $\begin{bmatrix} 3 & 1 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$. Replacement operations reduce this matrix to $\begin{bmatrix} 3 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$, which cofactor expansion along the first column shows has determinant 3. We deduce that $\det(A) = 90$. For (ii), let $B = \begin{bmatrix} 0 & 5 & 0 & 15 \\ 0 & 2 & 0 & 4 \\ -1 & 2 & 1 & 4 \\ 6 & 6 & 0 & 3 \end{bmatrix}$. By similar methods to (i), we compute $\det(B) = -60$. Cramer's rule then shows that $x_1 = -60/90 = -1/3$.