Received: 5 July 2022 | Revised: 24 October 2022 | Accepted: 22 November 2022

DOI: 10.1111/sapm.12551

ORIGINAL ARTICLE

STUDIES IN APPLIED MATHEMATICS WILEY

The 2D magnetohydrodynamic system with Euler-like velocity equation and partial magnetic diffusion

Tiantian Chen[1,2] | **Hongxia Lin**[1,2] | **Jiahong Wu**[3]

[1]Geomathematics Key Laboratory of Sichuan Province, Chengdu University of Technology, Chengdu, China

[2]College of Mathematics and Physics, Chengdu University of Technology, Chengdu, China

[3]Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma, USA

Correspondence
Hongxia Lin, Geomathematics Key Laboratory of Sichuan Province and College of Mathematics and Physics, Chengdu University of Technology, Chengdu, 610059, China.
Email: linhongxia18@126.com

Funding information
AT&T Foundation at Oklahoma State University; National Natural Science Foundation of China, Grant/Award Number: 11701049; National Science Foundation of the United States, Grant/Award Number: DMS2104682; the Natural Science Foundation of Sichuan Province, Grant/Award Number: 2023NSFSC0056

[Correction added on 12 January 2023, after first online publication: grant number corrected.]

Abstract

When the velocity equation of the incompressible 2D magnetohydrodynamic (MHD) system is inviscid, the global well-posedness and stability problem in the whole space $\mathbb{R}^2$ case remains an extremely challenging open problem. Broadman, Lin, and Wu (*SIAM J. Math. Anal.* 52(5) (2020): 5001-5035) were able to establish the global well-posedness and stability near a background magnetic field when there is damping in one velocity component. Their work exploited the stabilizing effect of the background magnetic field. This paper presents new progress. We are able to prove the global well-posedness and stability even when the magnetic diffusion is degenerate and only in the vertical direction. The velocity equation is still inviscid and has damping only in the vertical component. The proof of this new result overcomes two main difficulties, the potential rapid growth of the velocity due to the lack of dissipation or horizontal damping and the control of nonlinearity associated with the magnetic field. By discovering the key hidden smoothing effects and incorporating them in the construction of a two-layered energy function, we are able to obtain uniform bounds on the solution in the H^3-norm when the initial perturbation is near the background magnetic field. In addition, we prove that certain

Lebesgue and Sobolev norms of the solution approach zero as time approaches infinity.

KEYWORDS
background magnetic field, magnetohydrodynamic equation, partial dissipation, stability

1 | INTRODUCTION

The incompressible magnetohydrodynamic (MHD) system governs the motion of electrically conducting fluids in the presence of a magnetic field such as plasmas, liquid metals, and electrolytes (see, e.g., Refs. 1, 2). The standard incompressible MHD system can be written as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla P = \nu\Delta u + (B\cdot\nabla)B, \\ \partial_t B + (u\cdot\nabla)B = \eta\,\Delta B + (B\cdot\nabla)u, \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{1}$$

where u, B, and P represent the velocity field of the fluid, the magnetic field, and the scalar pressure, respectively. The parameters $\nu \geq 0$ and $\eta \geq 0$ denote the kinematic viscosity coefficient and the magnetic diffusivity, respectively.

In many physical applications, the MHD system with only partial dissipation is relevant. One especially important partial dissipation case is the MHD system with only magnetic diffusivity (resistivity). The fluid viscosity can be ignored while the role of resistivity is important in magnetic reconnection and magnetic turbulence. Magnetic reconnection refers to the breaking and reconnecting of oppositely directed magnetic field lines in a plasma and is at the heart of many spectacular events in our solar system (see, e.g., Ref. 3). Mathematically the 2D MHD system with only magnetic diffusivity is extremely difficult to analyze. In fact, fundamental issues such as the problem of whether or not solutions of the 2D resistive MHD system can develop finite time singularities remain open.

Even the small data global well-posedness problem concerning the 2D resistive MHD system is highly nontrivial. A recent brilliant work of Wei and Zhang[4] was able to solve the small data global well-posedness problem when the spatial domain is periodic and when the initial magnetic field has mean zero. When the spatial domain is the whole space $\mathbb{R}^2$, the small data global well-posedness problem remains open. Many efforts have been devoted to the 2D resistive MHD system in $\mathbb{R}^2$ and the well-posedness problem is now much better understood (see, e.g., Refs. 5–10).

Recently Broadman, Lin, and Wu[11] examined a special resistive MHD system with distinguishing mathematical properties. Besides the magnetic diffusion, the vertical component of the velocity involves a damping term. More precisely the MHD system considered in Ref. 11 assumes the form

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 = -\partial_1 P + (B\cdot\nabla)B_1, \quad x\in\mathbb{R}^2,\ t>0, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \gamma u_2 = -\partial_2 P + (B\cdot\nabla)B_2, \\ \partial_t B + u\cdot\nabla B = \eta\,\Delta B + B\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{2}$$

where $\gamma > 0$ is the damping coefficient and $\eta > 0$ the magnetic diffusivity. The velocity equation is an Euler-like equation and its solution can potentially grow in time. Ref. 11 was able to establish the global well-posedness and stability near a background magnetic field. The smoothing and stabilizing effect of the background magnetic field is fully exploited to overcome the potential growth in the velocity field.

This paper presents new progress. We are able to deal with the situation when the magnetic diffusion is degenerate and only in the vertical direction, namely,

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 = -\partial_1 P + (B\cdot\nabla)B_1, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t u_2 + (u\cdot\nabla)u_2 + \gamma u_2 = -\partial_2 P + (B\cdot\nabla)B_2,\\ \partial_t B + u\cdot\nabla B = \eta\,\partial_2^2 B + B\cdot\nabla u,\\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{3}$$

where the parameters $\gamma > 0$ and $\eta > 0$. A special steady-state solution of (3) is given by the background magnetic field

$$(u^{(0)}, B^{(0)}) = (0, e_2), \quad e_2 := (0,1). \tag{4}$$

The perturbation (u, b) with $b = B - B^{(0)}$ satisfies

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 = -\partial_1 P + (b\cdot\nabla)b_1 + \partial_2 b_1, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t u_2 + (u\cdot\nabla)u_2 + \gamma u_2 = -\partial_2 P + (b\cdot\nabla)b_2 + \partial_2 b_2,\\ \partial_t b + u\cdot\nabla b = \eta\,\partial_2^2 b + b\cdot\nabla u + \partial_2 u,\\ \nabla\cdot u = \nabla\cdot b = 0,\\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x). \end{cases} \tag{5}$$

Our goal here is to understand the global existence and stability of (u, b) governed by (5). To solve the desired stability problem, we need to overcome two main difficulties. The first is the potential growth of the velocity field u. The velocity equation in (5) is an Euler-like equation and its vorticity formulation involves the Riesz transform

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \gamma\mathcal{R}_1^2\omega + (b\cdot\nabla)j + \partial_2 j, \qquad x\in\mathbb{R}^2,\ t>0,\\ u = \nabla^\perp\Delta^{-1}\omega, \end{cases} \tag{6}$$

where $\omega = \nabla\times u$, $j = \nabla\times b$, $\nabla^\perp = (-\partial_2, \partial_1)$, and $\mathcal{R}_1 = \partial_1(-\Delta)^{-\frac{1}{2}}$ represent the Riesz transform. Here, the fractional Laplacian operator can be defined through the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi). \tag{7}$$

$u = \nabla^\perp\Delta^{-1}\omega$ represents the Biot–Savart law recovering the velocity u from the vorticity. The global well-posedness for the 2D Euler equation

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = 0, \qquad x\in\mathbb{R}^2,\ t>0,\\ u = \nabla^\perp\Delta^{-1}\omega, \end{cases} \tag{8}$$

has been well-known (see, e.g., Refs. 12, 13). Especially the classical Yudovich theory[13] asserted that there exists a unique global weak solution $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for any initial data $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Because the Riesz transform is not bounded in L^∞, the approach in Ref. 13 no longer works for the Euler-like equation

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \gamma \mathcal{R}_1^2 \omega, & x \in \mathbb{R}^2, \ t > 0, \\ u = \nabla^\perp \Delta^{-1} \omega. \end{cases} \tag{9}$$

The global well-posedness of classical solutions to (9) remains a well-known open problem. In addition, solutions to (9) can potentially grow rather rapidly in time, so small data global well-posedness and stability on (9) are also unknown. Therefore, solutions to (6) can potentially grow in time and the stability problem on (6) is possible only if the magnetic field helps.

The second difficulty is due to the lack of horizontal magnetic diffusion. In the whole space $\mathbb{R}^2$, dissipation in only one direction in general is not enough to control the nonlinearity. We discover here that the nonlinear terms have special structures, which allow us to bound them suitably via anisotropic tools.

This paper is able to successfully resolve these two difficulties and establish the desired well-posedness and stability stated in the following theorem.

Theorem 1. *Assume $(u_0, b_0) \in H^3(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, there exists $\varepsilon = \varepsilon(\gamma, \eta) > 0$ such that, if*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon, \tag{10}$$

then (5) admits a unique global solution (u, b) satisfying

$$\|(u(t), b(t))\|_{H^3}^2 + \int_0^t \left(\|u_2(\tau)\|_{H^3}^2 + \|\partial_2 u(\tau)\|_{H^2}^2 + \|\partial_2 b(\tau)\|_{H^3}^2 \right) d\tau \le C\varepsilon^2 \tag{11}$$

for any $t > 0$ and some uniform constant C. In addition, for any $2 \le p < \infty, 2 < q < \infty$, as $t \to \infty$, we have

$$\begin{aligned} &\|u_2(t)\|_{L^2(\mathbb{R}^2)} \to 0, \quad \|\nabla u(t)\|_{W^{1,p}(\mathbb{R}^2)} \to 0, \quad \|\partial_2 b(t)\|_{W^{1,p}(\mathbb{R}^2)} \to 0, \\ &\|u(t)\|_{L^q(\mathbb{R}^2)} \to 0, \quad \|(u(t), b(t))\|_{W^{1,\infty}(\mathbb{R}^2)} \to 0. \end{aligned} \tag{12}$$

As aforementioned, without the presence of the magnetic field, the fluid velocity u can potentially grow rapidly in time. Mathematically the best upper bound for u in the Sobolev space H^3 from direct energy estimates depends on time double exponentially. Therefore, it is absolutely crucial to explore the stabilizing effect of the magnetic field on u. The idea here is to unearth good hidden structures in (5). We are able to convert (5) into a system of wave equations. To do so, we first apply the projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to eliminate the pressure to obtain

$$\partial_t u = -\mathbb{P}(0, u_2)^\top + \partial_2 b - \mathbb{P}((b \cdot \nabla)b - (u \cdot \nabla)u). \tag{13}$$

By the definition of $\mathbb{P}$,

$$\mathbb{P}(0,u_2)^\top = (0,u_2)^\top - \nabla\Delta^{-1}\nabla\cdot(0,u_2)^\top = \Delta^{-1}\partial_1^2 u = -\mathcal{R}_1^2 u. \tag{14}$$

Therefore, (5) can be written as

$$\begin{cases}\partial_t u = \gamma\mathcal{R}_1^2 u + \partial_2 b + M_1, \quad x\in\mathbb{R}^2,\ t>0,\\ \partial_t b = \eta\,\partial_2^2 b + \partial_2 u + M_2,\\ \nabla\cdot u = \nabla\cdot b = 0,\end{cases} \tag{15}$$

where M_1 and M_2 are the nonlinear terms,

$$M_1 = \mathbb{P}((b\cdot\nabla)b - (u\cdot\nabla)u), \quad M_2 = (b\cdot\nabla)u - (u\cdot\nabla)b. \tag{16}$$

Differentiating (15) in time and making suitable substitutions, we obtain

$$\begin{cases}\partial_{tt}u - (\eta\partial_{22} + \gamma\mathcal{R}_1^2)\partial_t u + (\gamma\eta\partial_{22}\mathcal{R}_1^2 u - \partial_{22}u) = M_3,\\ \partial_{tt}b - (\eta\partial_{22} + \gamma\mathcal{R}_1^2)\partial_t b + (\gamma\eta\partial_{22}\mathcal{R}_1^2 b - \partial_{22}b) = M_4,\\ \nabla\cdot u = \nabla\cdot b = 0,\end{cases} \tag{17}$$

where M_3 and M_4 represent the nonlinear terms. In comparison with (5), (17) exhibits much more regularization. In particular, the term $\partial_{22}u$ generates some smoothing effect in the x_2-direction. A more elaborate examination on the wave structure would reveal that the smoothing effect in the x_2-direction is one derivative lower than the standard dissipation. More precisely, when we seek solutions in (5) in H^3, the enhanced dissipation in the x_2-direction is of the form $\|\partial_2 u\|_{H^2}$ while the standard dissipation would yield $\|\partial_2 u\|_{H^3}$. This fact motivates us to construct an energy functional with two parts, one reflecting the existing damping and dissipation of (5) and one incorporating the enhanced dissipation. To be exact, we define the energy functional E by

$$E(t) = E_1(t) + E_2(t), \tag{18}$$

where

$$E_1(t) = \sup_{0\le\tau\le t}\left(\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\right) + \int_0^t \left(\gamma\|u_2(\tau)\|_{H^3}^2 + \eta\|\partial_2 b(\tau)\|_{H^3}^2\right) d\tau,$$

$$E_2(t) = \int_0^t \|\partial_2 u(\tau)\|_{H^2}^2\, d\tau. \tag{19}$$

The inclusion of these two parts in the energy functional helps overcome the first difficulty described above.

To address the second difficulty (due to the lack of horizontal magnetic diffusion), our observation is that, in the process of energy estimates, each of the three triple products originated from the nonlinear terms $b\cdot\nabla b$, $u\cdot\nabla b$, and $b\cdot\nabla u$ contains u and its derivatives as a component. The regularity of u due to damping and enhanced dissipation coupled with the vertical magnetic diffusion allows us to obtain suitable upper bounds. Technically this is not trivial. We distinguish derivatives in different directions and employ various anisotropic tools such as anisotropic Sobolev inequalities and anisotropic upper bounds for triple products.

Our main efforts are devoted to implementing the ideas and strategies outlined above to establish the desired global bounds on (u, b) in H^3. We use the bootstrapping argument. The center piece is the proof of the following energy inequality, for a constant C_0 and for any $t > 0$,

$$E(t) \leq C_0 E(0) + C_0 E^{\frac{3}{2}}(t). \tag{20}$$

The task of proving (20) is naturally divided into two parts, for positive constants C_1 through C_5

$$E_1(t) \leq C_1 E(0) + C_2 E^{\frac{3}{2}}(t), \tag{21}$$

$$E_2(t) \leq C_3 E_1(0) + C_4 E_1(t) + C_5 E^{\frac{3}{2}}(t). \tag{22}$$

Once (20) is established, an application of the bootstrap argument (see Ref. 14, p. 21) would imply the desired global stability. Clearly, the key part of the proof is to establish the estimates (21) and (22). Here, we implement a very efficient approach to prove (22). Clearly the proof of (22) relies on the aforementioned vertical smoothing and the interaction between u and b. To obtain (22), it suffices to bound

$$\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 d\tau \qquad \text{and} \qquad \int_0^t \|\partial_2 \nabla \omega(\tau)\|_{L^2}^2 d\tau. \tag{23}$$

To bound the first integral, we replace $\partial_2 u$ by the equation of b,

$$\partial_2 u = \partial_t b + u \cdot \nabla b - \eta\, \partial_2^2 b - b \cdot \nabla u \tag{24}$$

to write

$$\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 d\tau = \int_0^t (\partial_2 u, \partial_t b + u \cdot \nabla b - \eta\, \partial_2^2 b - b \cdot \nabla u) d\tau, \tag{25}$$

Here, (f, g) denotes the L^2-inner product of f and g. By further shifting the time derivative and invoking the velocity equation, we are able to control all the terms suitably. To bound the second time integral in (23), we invoke the equations of ω and j,

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \gamma\, \mathcal{R}_1^2 \omega + (b \cdot \nabla) j + \partial_2 j, \\ \partial_t j + (u \cdot \nabla) j = \eta\, \partial_2^2 j + (b \cdot \nabla)\omega + \partial_2 \omega + Q, \end{cases} \tag{26}$$

where

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2). \tag{27}$$

In addition, the second equation in (26) is used to convert $\|\partial_2 \nabla \omega(\tau)\|_{L^2}^2$ into the inner product,

$$\|\partial_2 \nabla \omega\|_{L^2}^2 = (\partial_2 \nabla \omega, \partial_t \nabla j + \nabla(u \cdot \nabla) j - \eta \partial_2^2 \nabla j - \nabla(b \cdot \nabla)\omega - \nabla Q). \tag{28}$$

The time derivative on ∇j is further shifted to $\partial_2 \nabla \omega$ and the equation of ω is invoked. More details will be provided in Section 2.

We remark that the MHD system has been extensively studied since it was initially derived by H. Alfvén.[15] The well-posedness problem was first examined by the pioneering work of Duvaut and Lions,[16] and of Sermange and Temam.[17] There have been substantial recent developments. Significant progress has been made on the well-posedness and stability problems concerning various MHD systems such as those partial or fractional dissipation (see, e.g., Refs. 5–7, 9–11, 18–56). This list of references is by no means exhaustive.

We finally list several basic facts to be used in the subsequent sections,

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \quad \|\nabla^2 u\|_{L^2} = \|\nabla\omega\|_{L^2}, \quad \|\nabla^3 u\|_{L^2} = \|\nabla^2\omega\|_{L^2},$$
$$\|\nabla b\|_{L^2} = \|j\|_{L^2}, \quad \|\nabla^2 b\|_{L^2} = \|\nabla j\|_{L^2}, \quad \|\nabla^3 b\|_{L^2} = \|\nabla^2 j\|_{L^2}. \tag{29}$$

In addition, we use the norm notation $\|(u, b)\|_X^2$ for $\|u\|_X^2 + \|b\|_X^2$.

The rest of the paper is divided into two sections. Section 2 proves the global well-posedness part of Theorem 1. Most of the efforts are devoted to the proof of (30), which is naturally divided into two parts, the bound (21) for $E_1(t)$ and the bound (22) for $E_2(t)$. Section 3 focuses on proving the large time asymptotic behavior of the global solution.

2 | PROOF OF THE GLOBAL WELL-POSEDNESS

This section proves the global well-posedness part of Theorem 1. The procedure of proving the global well-posedness is to first construct a sequence of global and smooth approximate solutions, and then establish the global bounds that are uniform for the sequence, and finally extract a convergent subsequence with its limit solving (5). Because the construction of the approximating sequence and the limiting process are more or less standard, the task is reduced to the proof of the global bounds on solutions of (5). We only provide the proof for the global bounds.

We use the bootstrapping argument. The center piece is the estimate

$$E(t) \le C_0 E(0) + C_0 E^{\frac{3}{2}}(t). \tag{30}$$

The proof of (30) is naturally divided into two parts stated in the following two propositions.

Proposition 1. *Suppose that (u, b) is the solution of the system (5). Then it holds,*

$$E_1(t) \le C_1 E(0) + C_2 E^{\frac{3}{2}}(t) \tag{31}$$

for two positive constants C_1, C_2.

Proposition 2. *Let (u, b) be the solution of the system (5). Then there exist constants $C_3, C_4, C_5 > 0$ such that*

$$E_2(t) \le C_3 E_1(0) + C_4 E_1(t) + C_5 E^{\frac{3}{2}}(t). \tag{32}$$

The rest of this section is organized as follows. We first assume (30) and prove the well-posedness part of Theorem 1. We then prove Proposition 1 and Proposition 2.

Proof of the global well-posedness part in Theorem 1. The bootstrapping argument starts with an ansatz that $E(t)$ is bounded by

$$E(t) \le \frac{1}{4C_0^2}. \tag{33}$$

Next we show that $E(t)$ actually admits a smaller bound,

$$E(t) \le \frac{1}{8C_0^2}, \tag{34}$$

provided that the initial data are sufficiently small. Then, the bootstrap argument asserts that (34) holds for any time $t > 0$. To prove (34), we use (30). Inserting (33) in (30) yields

$$E(t) \le C_0 E(0) + \frac{1}{2}E(t) \quad \text{or} \quad E(t) \le 2C_0 E(0). \tag{35}$$

If we choose $\varepsilon > 0$ sufficiently small such that

$$\varepsilon \le \sqrt{\frac{1}{16C_0^3}}, \tag{36}$$

then it follows from (35) and (10) that

$$E(t) \le \frac{1}{8C_0^2}, \tag{37}$$

which is (34). This proves the global well-posedness part in Theorem 1. ■

The rest of this section proves Propositions 1 and 2. We first recall a powerful tool for the estimates of the nonlinear terms. It provides an anisotropic upper bound for a typical triple product. The proof of this lemma can be found in Ref. 5.

Lemma 1. *Assume $f, g, h, \partial_1 g, \partial_2 h \in L^2(\mathbb{R}^2)$. Then, for a constant $C > 0$,*

$$\iint fgh\, dx_1\, dx_2 \le C\|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\partial_1 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\partial_2 h\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}. \tag{38}$$

Proof of Proposition 1. Due to the equivalence

$$\|v\|_{H^3}^2 \sim \|v\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2, \tag{39}$$

we only need to bound $\|(u, b)\|_{L^2}$ and $\|(\Delta\omega, \Delta j)\|_{L^2}$. Taking the L^2-inner product of the system (5) with (u, b), integrating by parts and using the divergence free condition, we easily get

$$\|(u, b)(t)\|_{L^2}^2 + 2\int_0^t (\gamma\|u_2(\tau)\|_{L^2}^2 + \eta\|\partial_2 b(\tau)\|_{L^2}^2) d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{40}$$

Applying Δ to the system (26) and taking the L^2-inner product of the resulting equations with $(\Delta\omega, \Delta j)$ yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\Delta\omega,\Delta j)\|_{L^2}^2+\gamma\|\partial_1\nabla\omega\|_{L^2}^2+\eta\|\partial_2\Delta j\|_{L^2}^2\\
&=-\int \Delta(u\cdot\nabla)\omega\ \cdot\Delta\omega\,dx+\int\Delta(b\cdot\nabla)j\cdot\Delta\omega\,dx-\int\Delta(u\cdot\nabla)j\ \cdot\Delta j\,dx\\
&\quad+\int\Delta(b\cdot\nabla)\omega\cdot\Delta j\,dx+\int\Delta Q\cdot\Delta j\,dx:=I_1+I_2+\cdots+I_5,
\end{aligned} \tag{41}$$

where we have used $\int \Delta\mathcal{R}_1^2\omega\,\Delta\omega\,dx=-\|\Delta\mathcal{R}_1\omega\|_{L^2}^2=-\|\partial_1\nabla\omega\|_{L^2}^2$. Applying Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}
I_1&=-\int(\Delta u\cdot\nabla\omega)\cdot\Delta\omega\,dx-2\int\nabla u\cdot\nabla(\nabla\omega)\cdot\Delta\omega\,dx\\
&\le\|\Delta u\|_{L^4}\|\nabla\omega\|_{L^4}\|\Delta\omega\|_{L^2}+2\|\nabla u\|_{L^\infty}\|\nabla^2\omega\|_{L^2}^2\\
&\le C\|\Delta u\|_{H^1}\|\nabla\omega\|_{H^1}\|\Delta\omega\|_{L^2}+C\|\nabla u\|_{H^2}\|\nabla^2\omega\|_{L^2}^2\\
&\le C\|\nabla u\|_{H^2}\|\nabla\omega\|_{H^1}^2.
\end{aligned} \tag{42}$$

We bound I_2 and I_4 together. By means of the integration by parts and the divergence-free condition, the terms involving three derivatives in I_2 and I_4 are canceled.

$$\begin{aligned}
I_2+I_4=&\int(\Delta b\cdot\nabla j)\cdot\Delta\omega\,dx+2\int\nabla b\cdot\nabla(\nabla j)\cdot\Delta\omega\,dx\\
&+\int(\Delta b\cdot\nabla\omega)\cdot\Delta j\,dx+2\int\nabla b\cdot\nabla(\nabla\omega)\cdot\Delta j\,dx\\
:=&I_{21}+I_{22}+I_{23}+I_{24}.
\end{aligned} \tag{43}$$

It follows from Hölder's inequality and Sobolev's inequality that

$$\begin{aligned}
I_{21}&=\int\partial_1^2b_1\,\partial_1 j\,\partial_1^2\omega\,dx+\int\partial_1^2b_2\,\partial_2 j\,\partial_1^2\omega\,dx+\int\partial_2^2b\cdot\nabla j\,\partial_2^2\omega\,dx\\
&\le\|\partial_1^2b_2\|_{L^4}\|\partial_2 j\|_{L^4}\|\partial_1^2\omega\|_{L^2}+C\|\partial_2\nabla b\|_{L^4}\|\nabla j\|_{L^4}(\|\partial_1^2\omega\|_{L^2}+\|\partial_2^2\omega\|_{L^2})\\
&\le C\|\nabla j\|_{H^1}(\|\Delta\omega\|_{L^2}^2+\|\partial_2 j\|_{H^1}^2)
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
I_{22}&=2\int\partial_1b_1\,\partial_1^2 j\,\partial_1^2\omega\,dx+2\int\partial_1b_2\,\partial_1\partial_2 j\,\partial_1^2\omega\,dx+2\int\partial_2 b\cdot\nabla(\partial_2 j)\,\partial_2^2\omega\,dx\\
&\le C\|\partial_2 b\|_{H^2}\|\partial_1^2 j\|_{L^2}\|\partial_1^2\omega\|_{L^2}+C\|\nabla b\|_{H^2}\|\nabla\partial_2 j\|_{L^2}(\|\partial_1^2\omega\|_{L^2}+\|\partial_2^2\omega\|_{L^2})\\
&\le C\|j\|_{H^2}(\|\Delta\omega\|_{L^2}^2+\|\partial_2 b\|_{H^2}^2).
\end{aligned} \tag{45}$$

For I_{23}, we first split it into three parts and integrate by parts in the second part to get

$$
\begin{aligned}
I_{23} &= \int \partial_1^2 b_1\, \partial_1\omega\, \partial_1^2 j\, dx - \int \omega\, (\partial_1^2\partial_2 b_2\, \partial_1^2 j + \partial_1^2 b_2\, \partial_1^2\partial_2 j)\, dx + \int \partial_2^2 b \cdot \nabla\omega\, \partial_2^2 j\, dx \\
&\le C\|\partial_2 \nabla b\|_{L^4}\|\nabla\omega\|_{L^4}(\|\partial_1^2 j\|_{L^2} + \|\partial_2^2 j\|_{L^2}) \\
&\quad + \|w\|_{L^\infty}\left(\|\partial_1^2\partial_2 b_2\|_{L^2}\|\partial_1^2 j\|_{L^2} + \|\partial_1^2 b_2\|_{L^2}\|\partial_1^2\partial_2 j\|_{L^2}\right) \\
&\le C\|\nabla j\|_{H^1}(\|\omega\|_{H^2}^2 + \|\partial_2 j\|_{H^2}^2).
\end{aligned} \tag{46}
$$

Similarly,

$$
\begin{aligned}
I_{24} =& 2\int \partial_1 b_1 \partial_1^2\omega\, \partial_1^2 j\, dx - 2\int \partial_1\omega\, (\partial_1\partial_2 b_2\, \partial_1^2 j + \partial_1 b_2\, \partial_1^2\partial_2 j)\, dx \\
&+ 2\int \partial_2 b \cdot \nabla(\partial_2\omega)\, \partial_2^2 j\, dx \\
\le& C\|\partial_2 b\|_{H^2}\|\nabla^2\omega\|_{L^2}\|\nabla^2 j\|_{L^2} + C\|\partial_1\omega\|_{L^2}\left(\|\partial_1\partial_2 b\|_{H^2}\|\partial_1^2 j\|_{L^2} + \|\partial_1 b\|_{H^2}\|\partial_1^2\partial_2 j\|_{L^2}\right) \\
\le& C\|j\|_{H^2}(\|\nabla\omega\|_{H^1}^2 + \|\partial_2 b\|_{H^3}^2).
\end{aligned} \tag{47}
$$

Consequently,

$$
I_2 + I_4 \le C\|j\|_{H^2}(\|\omega\|_{H^2}^2 + \|\partial_2 b\|_{H^3}^2). \tag{48}
$$

We turn back to estimate I_3. First,

$$
I_3 = -\int (\Delta u \cdot \nabla j) \cdot \Delta j\, dx - 2\int \nabla u \cdot \nabla(\nabla j) \cdot \Delta j\, dx. \tag{49}
$$

For the first part of I_3, by the incompressible condition $\partial_1 u_1 = -\partial_2 u_2$ and the integration by parts, we further split it in three parts to get

$$
\begin{aligned}
-\int (\Delta u \cdot \nabla j) \cdot \Delta j\, dx =& -\int \partial_1^2 u_1 \partial_1 j\, \partial_1^2 j\, dx - \int \partial_1^2 u_2\, \partial_2 j\, \partial_1^2 j\, dx - \int \partial_2^2 u \cdot \nabla j\, \partial_2^2 j\, dx \\
=& \int \partial_1\partial_2 u_2 \partial_1 j\, \partial_1^2 j\, dx - \int \partial_1^2 u_2\, \partial_2 j\, \partial_1^2 j\, dx - \int \partial_2^2 u \cdot \nabla j\, \partial_2^2 j\, dx \\
=& -\int \partial_1 u_2 (\partial_1\partial_2 j\, \partial_1^2 j + \partial_1 j\, \partial_1^2\partial_2 j)\, dx - \int \partial_1^2 u_2\, \partial_2 j\, \partial_1^2 j\, dx \\
&- \int \partial_2^2 u \cdot \nabla j\, \partial_2^2 j\, dx.
\end{aligned} \tag{50}
$$

Then, Hölder's inequality and Sobolev's inequality yield

$$
\begin{aligned}
-\int (\Delta u \cdot \nabla j) \cdot \Delta j\, dx \le& \|\partial_1 u_2\|_{L^\infty}(\|\partial_1\partial_2 j\|_{L^2}\|\partial_1^2 j\|_{L^2} + \|\partial_1 j\|_{L^2}\|\partial_1^2\partial_2 j\|_{L^2}) \\
&+ \|\partial_1^2 u\|_{L^4}\|\partial_2 j\|_{L^4}\|\partial_1^2 j\|_{L^2} + \|\partial_2^2 u\|_{L^4}\|\nabla j\|_{L^2}\|\partial_2^2 j\|_{L^4} \\
\le& C\|\nabla j\|_{H^1}(\|\nabla u\|_{H^2}^2 + \|\partial_2 j\|_{H^2}^2).
\end{aligned} \tag{51}
$$

Similarly, we have

$$-2\int \nabla u\cdot\nabla(\nabla j)\cdot\Delta j\,dx = -4\int u_2\,\partial_1^2 j\,\partial_1^2\partial_2 j - 2\int \partial_1 u_2\partial_1\partial_2 j\,\partial_1^2 j\,dx$$
$$-2\int \partial_2 u\cdot\nabla\partial_2 j\,\partial_2^2 j\,dx$$
$$\leq C\|u_2\|_{H^2}\|\partial_1^2 j\|_{L^2}\|\partial_1^2\partial_2 j\|_{L^2} + C\|\nabla u\|_{H^1}\|\nabla\partial_2 j\|_{H^1}\|\nabla^2 j\|_{L^2}$$
$$\leq C\|\nabla^2 j\|_{L^2}(\|u_2\|_{H^2}^2 + \|\nabla u\|_{H^1}^2 + \|\partial_2 j\|_{H^2}^2). \tag{52}$$

Thus, we obtain

$$I_3 \leq C\|\nabla j\|_{H^1}(\|u_2\|_{H^2}^2 + \|\nabla u\|_{H^2}^2 + \|\partial_2 j\|_{H^2}^2). \tag{53}$$

Now we deal with the last term I_5. According to the definition of Q, we divide it into two parts

$$I_5 = 2\int \Delta(\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2))\cdot\Delta j\,dx - 2\int \Delta(\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2))\cdot\Delta j\,dx := I_{51} + I_{52}. \tag{54}$$

By Hölder's and Sobolev's inequalities,

$$I_{51} \leq 2\int |\Delta\partial_2 b_2|\,|\nabla u|\,|\Delta j|\,dx + 4\int |\nabla\partial_2 b_2|\,|\nabla^2 u|\,|\Delta j|\,dx + 2\int |\partial_2 b_2|\,|\nabla^3 u|\,|\Delta j|\,dx$$
$$\leq C\big(\,\|\Delta\partial_2 b_2\|_{L^2}\|\nabla u\|_{H^2} + \|\nabla\partial_2 b_2\|_{H^1}\|\nabla^2 u\|_{H^1} + \|\partial_2 b_2\|_{H^2}\|\nabla^3 u\|_{L^2}\,\big)\,\|\Delta j\|_{L^2}$$
$$\leq C\|\Delta j\|_{L^2}(\|\partial_2 b\|_{H^2}^2 + \|\nabla u\|_{H^2}^2). \tag{55}$$

For I_{52}, we first split it into three parts and then bound it one by one.

$$I_{52} = -2\int \Delta\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)\cdot\Delta j\,dx - 4\int \nabla\partial_1 u_1\cdot(\nabla\partial_2 b_1 + \nabla\partial_1 b_2)\cdot\Delta j\,dx$$
$$-2\int \partial_1 u_1(\Delta\partial_2 b_1 + \Delta\partial_1 b_2)\cdot\Delta j\,dx$$
$$=: I_{52,1} + I_{52,2} + I_{52,3}. \tag{56}$$

By the divergence-free condition for u, integration by parts, Hölder's inequality, and Sobolev's inequality, we derive

$$I_{52,1} = -2\int \Delta u_2\big[(\partial_2^2 b_1 + \partial_1\partial_2 b_2)\cdot\Delta j + (\partial_2 b_1 + \partial_1 b_2)\cdot\Delta\partial_2 j\big]\,dx$$
$$\leq C\|\Delta u_2\|_{L^2}(\|\partial_2\nabla b\|_{H^2}\|\Delta j\|_{L^2} + \|\nabla b\|_{H^2}\|\Delta\partial_2 j\|_{L^2})$$
$$\leq C\|j\|_{H^2}(\|\Delta u_2\|_{L^2}^2 + \|\partial_2 j\|_{H^2}^2). \tag{57}$$

Similarly, $I_{52,2}$ and $I_{52,3}$ can be bounded by

$$I_{52,2} = -4\int \nabla u_2 \cdot \left[(\nabla\partial_2^2 b_1 + \nabla\partial_1\partial_2 b_2)\cdot\Delta j + (\nabla\partial_2 b_1 + \nabla\partial_1 b_2)\cdot\Delta\partial_2 j\right]dx$$
$$\leq C\|j\|_{H^2}(\|\nabla u_2\|_{H^2}^2 + \|\partial_2 j\|_{H^2}^2),$$
$$I_{52,3} = -2\int u_2\left[(\Delta\partial_2^2 b_1 + \Delta\partial_1\partial_2 b_2)\cdot\Delta j + (\Delta\partial_2 b_1 + \Delta\partial_1 b_2)\cdot\Delta\partial_2 j\right]dx$$
$$\leq C\|j\|_{H^2}(\|u_2\|_{H^2}^2 + \|\partial_2 j\|_{H^2}^2). \tag{58}$$

Collecting all the estimates above yields

$$I_5 \leq C\|j\|_{H^2}(\|u_2\|_{H^3}^2 + \|\nabla u\|_{H^2}^2 + \|\partial_2 j\|_{H^2}^2). \tag{59}$$

Inserting (42), (48), (53), and (59) in (41), we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\Delta\omega,\Delta j)(t)\|_{L^2}^2 + \gamma\|\partial_1\nabla\omega\|_{L^2}^2 + \eta\|\partial_2\Delta j\|_{L^2}^2$$
$$\leq C(\|\omega\|_{H^2} + \|j\|_{H^2})(\|u_2\|_{H^3}^2 + \|\nabla u\|_{H^2}^2 + \|\partial_2 b\|_{H^3}^2). \tag{60}$$

Then, integrating it on $[0, t]$, we derive

$$\|(\Delta\omega,\Delta j)\|_{L^2}^2 + 2\int_0^t(\gamma\|\partial_1\nabla\omega(\tau)\|_{L^2}^2 + \eta\|\partial_2\Delta j(\tau)\|_{L^2}^2)\,d\tau$$
$$\leq (\|\Delta\omega_0\|_{L^2}^2 + \|\Delta j_0\|_{L^2}^2) + CE^{\frac{3}{2}}(t). \tag{61}$$

Recalling (40), we conclude

$$\|(u,b)\|_{H^3}^2 + \int_0^t(\gamma\|u_2(\tau)\|_{H^3}^2 + \eta\|\partial_2 b(\tau)\|_{H^3}^2)\,d\tau \leq CE(0) + CE^{\frac{3}{2}}(t), \tag{62}$$

where we have used $\|\partial_1\nabla\omega\|_{L^2}^2 = \|\nabla^3 u_2\|_{L^2}^2$. This ends the proof of Proposition 1. ■

Next we prove Proposition 2, which provides the estimate on $\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau$.

Proof of Proposition 2. Because the norm $\|v\|_{H^2}^2$ is equivalent to $\|v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2$, it is sufficient to bound

$$\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 d\tau \qquad \text{and} \qquad \int_0^t \|\partial_2\nabla\omega(\tau)\|_{L^2}^2 d\tau, \tag{63}$$

where we have used the fact $\|\partial_2\nabla\omega\|_{L^2} = \|\partial_2\Delta u\|_{L^2}$. By the equations in (5), we have

$$\frac{d}{dt}(\partial_2 u, b) = (\partial_2 u_t, b) + (\partial_2 u, b_t)$$
$$= (\partial_2(-u\cdot\nabla u - (0,\gamma u_2) - \nabla P + b\cdot\nabla b + \partial_2 b),\, b)$$
$$+ \left(\partial_2 u,\, -u\cdot\nabla b + \eta\,\partial_2^2 b + b\cdot\nabla u + \partial_2 u\right), \tag{64}$$

where (A, B) stands for the L^2-inner product of A and B. Thereby, we infer

$$
\begin{aligned}
&-\frac{d}{dt}(\partial_2 u, b) + \|\partial_2 u\|_{L^2}^2 - \|\partial_2 b\|_{L^2}^2 \\
&= \int (\partial_2(u\cdot\nabla u)\cdot b - \partial_2 u\cdot(b\cdot\nabla u))\,dx \\
&\quad + \int (-\partial_2(b\cdot\nabla b)\cdot b + \partial_2 u\cdot(u\cdot\nabla b))\,dx \\
&\quad + \int (\gamma\partial_2 u_2\, b_2 - \eta\,\partial_2 u\cdot\partial_2^2 b)\,dx. \qquad (65)
\end{aligned}
$$

By integration by parts, Hölder's inequality and Sobolev's inequality, we obtain

$$
\begin{aligned}
&\int (\partial_2(u\cdot\nabla u)\cdot b - \partial_2 u\cdot(b\cdot\nabla u))\,dx \\
&= -\int ((u\cdot\nabla u)\cdot\partial_2 b + \partial_2 u\cdot(b\cdot\nabla u))\,dx \\
&\le \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\partial_2 b\|_{L^2} + \|b\|_{L^\infty}\|\nabla u\|_{L^2}^2 \\
&\le C(\|u\|_{H^2} + \|b\|_{H^2})(\|\nabla u\|_{L^2}^2 + \|\partial_2 b\|_{L^2}^2). \qquad (66)
\end{aligned}
$$

The second part is first divided into three terms. Then, Lemma 1 together with Sobolev's inequality leads to

$$
\begin{aligned}
&\int ((-\partial_2(b\cdot\nabla b)\cdot b + \partial_2 u\cdot(u\cdot\nabla b))\,dx \\
&= \int (b_1\partial_1 b\cdot\partial_2 b + b_2\partial_2 b\cdot\partial_2 b + \partial_2 u\cdot(u\cdot\nabla b))\,dx \\
&\le C\|b_1\|_{L^2}^{\frac12}\|\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1\partial_2 b\|_{L^2}^{\frac12}\|\partial_2 b\|_{L^2} + C\|b\|_{H^2}\|\partial_2 b\|_{L^2}^2 \\
&\quad + C\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2} \\
&\le C(\|u\|_{L^2} + \|b\|_{H^2})(\|\nabla u\|_{L^2}^2 + \|\partial_2 b\|_{H^1}^2). \qquad (67)
\end{aligned}
$$

Clearly, for the last part, we have

$$
\begin{aligned}
&\int \left(\gamma\partial_2 u_2\, b_2 - \eta\partial_2 u\cdot\partial_2^2 b\right)dx = -\int \left(\gamma u_2\,\partial_2 b_2 + \eta\partial_2 u\cdot\partial_2^2 b\right)dx \\
&\le \frac{\gamma}{2}\|u_2\|_{L^2}^2 + \frac{\gamma}{2}\|\partial_2 b_2\|_{L^2}^2 + \frac{\eta^2}{2}\|\partial_2^2 b\|_{L^2}^2 + \frac12\|\partial_2 u\|_{L^2}^2. \qquad (68)
\end{aligned}
$$

Combining all the estimates above and integrating in time, we obtain

$$\begin{aligned}\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 d\tau \leq & \Big(\|\partial_2 u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\partial_2 u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\Big) \\ & + C \sup_{0\leq\tau\leq t} (\|u(\tau)\|_{H^2} + \|b(\tau)\|_{H^2}) \int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \|\partial_2 b(\tau)\|_{H^1}^2) d\tau \\ & + \int_0^t \Big(\gamma \|u_2(\tau)\|_{L^2}^2 + (\gamma+2)\|\partial_2 b(\tau)\|_{L^2}^2 + \eta^2 \|\partial_2^2 b(\tau)\|_{L^2}^2\Big) d\tau \\ \leq & E_1(0) + CE_1(t) + CE^{\frac{3}{2}}(t),\end{aligned} \tag{69}$$

where we have used the bound

$$\int_0^t \frac{d}{d\tau}(\partial_2 u, b) d\tau \leq \frac{1}{2}\Big(\|\partial_2 u\|_{L^2}^2 + \|b\|_{L^2}^2\Big) + \frac{1}{2}\Big(\|\partial_2 u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\Big). \tag{70}$$

Next, we present the estimate for $\int_0^t \|\partial_2 \nabla\omega(\tau)\|_{L^2}^2 d\tau$. We will make use of Equation (26) of (w, j) . By (26), a similar argument to (64) leads to

$$\begin{aligned}& -\frac{d}{dt}(\partial_2 \nabla\omega, \nabla j) + \|\partial_2 \nabla\omega\|_{L^2}^2 - \|\partial_2 \nabla j\|_{L^2}^2 \\ & = \int \partial_2 \nabla(u\cdot\nabla\omega)\cdot\nabla j\, dx - \int \partial_2\nabla(b\cdot\nabla j)\cdot\nabla j\, dx \\ & \quad + \int \partial_2\nabla\omega\cdot\nabla(u\cdot\nabla j)\, dx - \int \partial_2\nabla\omega\cdot\nabla(b\cdot\nabla\omega)\, dx \\ & \quad - \int \nabla Q\cdot\partial_2\nabla\omega\, dx - \gamma\int \partial_2\nabla\mathcal{R}_1^2\omega\cdot\nabla j\, dx - \eta\int \partial_2\nabla\omega\cdot\partial_2^2\nabla j\, dx \\ & = J_1 + J_2 + \cdots + J_7.\end{aligned} \tag{71}$$

Similarly to (66), J_1 and J_4 can be bounded as

$$\begin{aligned}J_1 + J_4 = & \int (u\cdot\nabla\omega)\,\partial_2\Delta j\, dx - \int \partial_2\nabla\omega\cdot(\nabla b\cdot\nabla\omega + (b\cdot\nabla)\nabla\omega)\, dx \\ \leq & C\|u\|_{H^2}\|\nabla\omega\|_{L^2}\|\partial_2\Delta j\|_{L^2} + C\big(\|\nabla b\|_{H^2}\|\nabla\omega\|_{L^2} + \|b\|_{H^2}\|\nabla^2\omega\|_{L^2}\big)\|\partial_2\nabla\omega\|_{L^2} \\ \leq & C(\|u\|_{H^2} + \|b\|_{H^3})(\|\nabla\omega\|_{H^1}^2 + \|\partial_2\Delta j\|_{L^2}^2).\end{aligned} \tag{72}$$

For J_2 , using the integration by parts and applying (38) yield

$$\begin{aligned}J_2 = & -\int (b_1\partial_1 j + b_2\partial_2 j)\,\partial_2\Delta j\, dx \\ \leq & C\|b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta j\|_{L^2} + C\|b\|_{H^2}\|\partial_2 j\|_{L^2}\|\partial_2\Delta j\|_{L^2} \\ \leq & C\|b\|_{H^2}\|\partial_2 b\|_{H^3}^2.\end{aligned} \tag{73}$$

Similarly, by (38) and Sobolev's inequality $\|v\|_{L^4(\mathbb{R}^2)} \le C\|v\|_{H^1(\mathbb{R}^2)}$

$$\begin{aligned}J_3 =& \int (\nabla u \cdot \nabla j + u \cdot \nabla(\nabla j)) \cdot \partial_2 \nabla \omega \, dx \\ \le& C\|\nabla u\|_{H^1}\|\nabla j\|_{H^1}\|\partial_2\nabla\omega\|_{L^2} + C\|u\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^2 j\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\omega\|_{L^2} \\ \le& C(\|u\|_{L^2} + \|\nabla j\|_{H^1})(\|\nabla u\|_{H^2}^2 + \|\partial_2\nabla^2 j\|_{L^2}^2). \end{aligned} \tag{74}$$

Due to the good form of Q , the bound for J_5 is simple.

$$\begin{aligned}J_5 =& -2\int [\partial_1 b_1(\nabla\partial_2 u_1 + \nabla\partial_1 u_2) - \partial_1 u_1(\nabla\partial_2 b_1 + \nabla\partial_1 b_2)] \cdot \partial_2\nabla\omega \, dx \\ & -2\int [\nabla\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - \nabla\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)] \cdot \partial_2\nabla\omega \, dx \\ \le& C\left(\|\nabla b\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}\right)\|\partial_2\nabla\omega\|_{L^2} \\ \le& C\|b\|_{H^3}\|\nabla u\|_{H^2}^2. \end{aligned} \tag{75}$$

The linear integrals can be directly estimated as

$$J_6 + J_7 \le \frac{\gamma}{2}(\|\nabla^2 u_2\|_{L^2}^2 + \|\partial_2\nabla j\|_{L^2}^2) + \frac{1}{2}\|\partial_2\nabla\omega\|_{L^2}^2 + \frac{\eta^2}{2}\|\partial_2^2\nabla j\|_{L^2}^2, \tag{76}$$

where we have used $\|\nabla\mathcal{R}_1^2\omega\|_{L^2} \le \|\partial_1\omega\|_{L^2} = \|\nabla^2 u_2\|_{L^2}$.

Inserting (72), (73), (74), (75), (76) in (77), integrating it on $[0, t]$, we derive

$$\begin{aligned}\int_0^t \|\partial_2\nabla\omega(\tau)\|_{L^2}^2 \, d\tau \le& \|(\partial_2\nabla\omega, \nabla j)\|_{L^2}^2 + \|(\partial_2\nabla\omega_0, \nabla j_0)\|_{L^2}^2 + 2\int_0^t \|\partial_2\nabla j(\tau)\|_{L^2}^2 \, d\tau \\ & + \gamma\int_0^t (\|\nabla^2 u_2(\tau)\|_{L^2}^2 + \|\partial_2\nabla j(\tau)\|_{L^2}^2)\, d\tau + \eta^2\int_0^t \|\partial_2^2\nabla j(\tau)\|_{L^2}^2 \, d\tau \\ & + C\int_0^t (\|u(\tau)\|_{H^2} + \|b(\tau)\|_{H^3})(\|\nabla u(\tau)\|_{H^2}^2 + \|\partial_2 b(\tau)\|_{H^3}^2)\, d\tau \\ \le& E_1(0) + CE_1(t) + CE^{\frac{3}{2}}(t), \end{aligned} \tag{77}$$

which together with (69) yields the desired bound (32). This completes the proof of Proposition 2. ■

Now we are ready to prove (30).

Proof (of (30)). Multiplying (32) by $\frac{1}{2C_4}$ and adding to (31), we infer

$$E(t) \le C_0 E(0) + C_0 E^{\frac{3}{2}}(t) \tag{78}$$

for a constant $C_0 > 0$. This completes the proof of (30). ■

3 | PROOF OF THE LONG TIME BEHAVIOR

This section is devoted to proving the long time behavior (12). We will employ the following lemma, which provides an easily verifiable condition under which a nonnegative and integrable function actually approaches zero at infinity. The proof can be found in Ref. 57, Lemma 3.1.

Lemma 2. *Assume $f = f(t)$ with $t \in [0, \infty)$ is a nonnegative and uniform continuous function satisfying*

$$\int_0^\infty f(t)\, dt < \infty. \tag{79}$$

Then

$$f(t) \to 0 \quad \text{as } t \to \infty. \tag{80}$$

Now we give the proof of (12).

Proof (of (12)). First, we have the following anisotropic Sobolev inequality

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq C\|v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_1 v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_2 v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_1\partial_2 v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}. \tag{81}$$

In fact, by Minkowski's inequality and the basic inequality for $i = 1, 2$

$$\|v\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|v\|^{\frac{1}{2}}_{L^2_{x_i}(\mathbb{R})}\|\partial_i v\|^{\frac{1}{2}}_{L^2_{x_i}(\mathbb{R})}, \tag{82}$$

we have

$$\begin{aligned}
\|v\|_{L^\infty(\mathbb{R}^2)} &= \big\|\|v\|_{L^\infty_{x_2}(\mathbb{R})}\big\|_{L^\infty_{x_1}(\mathbb{R})} \\
&\leq \sqrt{2}\|v\|^{\frac{1}{2}}_{L^2_{x_2}L^\infty_{x_1}}\|\partial_2 v\|^{\frac{1}{2}}_{L^2_{x_2}L^\infty_{x_1}} \\
&\leq C\|v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_1 v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_2 v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_1\partial_2 v\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{83}$$

Then, from (81) and the following Sobolev's inequality,

$$\|v\|_{L^p(\mathbb{R}^2)} \leq C\|v\|^{\frac{2}{p}}_{L^2(\mathbb{R}^2)}\|\nabla v\|^{1-\frac{2}{p}}_{L^2(\mathbb{R}^2)}, \quad \text{for} \quad 2 \leq p < \infty, \tag{84}$$

we claim that to prove (12), it suffices to prove the long time behavior of $\|u_2\|_{L^2}$, $\|\nabla u\|_{H^1}$ and $\|\partial_2 b\|_{H^1}$, that is,

$$\|u_2(t)\|_{L^2} \to 0, \quad \|\nabla u(t)\|_{H^1} \to 0 \quad \text{and} \quad \|\partial_2 b(t)\|_{H^1} \to 0 \quad \text{as } t \to \infty. \tag{85}$$

Invoking (11) and the fact $\|\nabla u_2\|_{H^1} = \|\partial_1 u\|_{H^1}$, we obtain

$$\int_0^\infty (\|u_2(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{H^1} + \|\partial_2 b(t)\|^2_{H^1})\, dt < \infty. \tag{86}$$

Thus, we just need to verify $\|u_2\|_{L^2}$, $\|\nabla u\|_{H^1}$, and $\|\partial_2 b\|_{H^1}$ satisfy the uniform continuity part of Lemma 2. Multiplying the equation of u_2 in (15) by u_2 and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_{L^2}^2 + \gamma\|\mathcal{R}_1 u_2\|_{L^2}^2 = -\int (\mathbb{P}(u\cdot\nabla u))_2\, u_2\, dx + \int (\mathbb{P}(b\cdot\nabla b))_2\, u_2\, dx + \int \partial_2 b_2\, u_2\, dx. \tag{87}$$

Recalling that $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ and applying Hölder's inequality, Sobolev's inequality, and the uniform bound (11) yield

$$\begin{aligned}\left|\int (\mathbb{P}(u\cdot\nabla u))_2\, u_2\, dx\right| &= \left|\int \Delta^{-1}\nabla\cdot(u\cdot\nabla u)\,\partial_2 u_2\, dx\right| \\ &= \left|\int \Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\,\partial_2 u_2 dx\right| \le \|u\otimes u\|_{L^2}\|\partial_2 u_2\|_{L^2} \\ &\le \|u\|_{L^4}^2\|\partial_2 u_2\|_{L^2} \le C\|u\|_{H^1}^3 \le C\varepsilon^3, \end{aligned}\tag{88}$$

where we have used the fact that the singular integral operator $\Delta^{-1}\nabla\cdot\nabla\cdot$ is bounded on L^2 (see[58]). Similarly,

$$\begin{aligned}\left|\int (\mathbb{P}(b\cdot\nabla b))_2\, u_2\, dx\right| &= \left|\int b\cdot\nabla b_2\, u_2\, dx + \int \Delta^{-1}\nabla\cdot(b\cdot\nabla b)\,\partial_2 u_2\, dx\right| \\ &\le \|b\|_{L^4}\|\nabla b\|_{L^2}\|u_2\|_{L^4} + \|b\|_{L^4}^2\|\partial_2 u_2\|_{L^2} \\ &\le C\|b\|_{H^1}^2\|u_2\|_{H^1} \le C\varepsilon^3. \end{aligned}\tag{89}$$

Clearly,

$$\left|\int \partial_2 b_2\, u_2\, dx\right| \le \frac{1}{2}(\|\partial_2 b_2\|_{L^2}^2 + \|u_2\|_{L^2}^2) \le C\varepsilon^2. \tag{90}$$

Combining the bounds above, we derive

$$\left|\frac{d}{dt}\|u_2(t)\|_{L^2}^2\right| \le C(\varepsilon), \tag{91}$$

where we have used the bound of the Riesz transform in $L^q(1 < q < \infty)$, namely,

$$\|\mathcal{R}_1 u_2\|_{L^2} \le \|u_2\|_{L^2} \le C\varepsilon. \tag{92}$$

Then, (91) implies the uniform continuity in Lemma 2. Consequently, we conclude

$$\|u_2(t)\|_{L^2} \to 0 \ \text{ as } \ t\to\infty. \tag{93}$$

We proceed to prove the uniform continuity for $\|\nabla u\|_{H^1}$. Taking the H^1-inner product of the ω equation in (26) with ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{H^1}^2 + \gamma\|\partial_1 u\|_{H^1}^2 = -(u\cdot\nabla\omega,\omega)_{H^1} + (b\cdot\nabla j,\omega)_{H^1} + (\partial_2 j,\omega)_{H^1}, \tag{94}$$

where we have used $(\mathcal{R}_1^2\omega,\omega)_{H^1} = -\|\mathcal{R}_1\omega\|_{H^1}^2 = -\|\partial_1 u\|_{H^1}^2$. By the integration by parts and incompressible condition, Hölder's inequality, and Sobolev's inequity, we infer

$$\begin{aligned} |(u\cdot\nabla\omega,\omega)_{H^1}| &= \Big|\int (\nabla u\cdot\nabla\omega)\cdot\nabla\omega\,dx\Big| \\ &\le \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^2}^2 \le C\|\nabla u\|_{H^2}\|\nabla\omega\|_{L^2}^2 \le C\varepsilon^3. \end{aligned} \tag{95}$$

Similarly,

$$\begin{aligned} |(b\cdot\nabla j,\omega)_{H^1}| &= \Big|\int (b\cdot\nabla j)\,\omega\,dx + \int (\nabla b\cdot\nabla j)\cdot\nabla\omega\,dx + \int (b\cdot\nabla)\nabla j\cdot\nabla\omega\,dx\Big| \\ &\le \|b\|_{L^\infty}(\|\nabla j\|_{L^2}\|\omega\|_{L^2} + \|\nabla^2 j\|_{L^2}\|\nabla\omega\|_{L^2}) + \|\nabla b\|_{L^\infty}\|\nabla j\|_{L^2}\|\nabla\omega\|_{L^2} \\ &\le C\|b\|_{H^3}\|\nabla j\|_{H^1}\|\omega\|_{H^1} \le C\varepsilon^3. \end{aligned} \tag{96}$$

Also,

$$|(\partial_2 j,\omega)_{H^1}| \le \frac{1}{2}(\|\partial_2 j\|_{H^1}^2 + \|\omega\|_{H^1}^2) \le C\varepsilon^2. \tag{97}$$

Therefore, we can conclude

$$\Big|\frac{d}{dt}\|\omega(t)\|_{H^1}^2 + 2\gamma\|\partial_1 u\|_{H^1}^2\Big| \le C(\varepsilon), \tag{98}$$

which together with $\|\partial_1 u\|_{H^1}^2 \le C\varepsilon^2$ yields the uniform continuity for $\|\nabla u(t)\|_{H^1}^2$. Hence, we get

$$\|\nabla u(t)\|_{H^1} \to 0 \ \text{ as } \ t\to\infty. \tag{99}$$

The uniform continuity for $\|\partial_2 b(t)\|_{H^1}^2$ can be obtained in a similar way. Applying the operator ∂_2 to the equation of b in (5), taking the H^1-inner product of the resulting equation with $\partial_2 b$, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_2 b(t)\|_{H^1}^2 + \eta\|\partial_2^2 b\|_{H^1}^2 = -(\partial_2(u\cdot\nabla b),\partial_2 b)_{H^1} + (\partial_2(b\cdot\nabla u),\partial_2 b)_{H^1} + (\partial_2^2 u,\partial_2 b)_{H^1}. \tag{100}$$

By Hölder's inequality and Sobolev's inequality, the three terms on the right can be bounded as:

$$\begin{aligned} |-(\partial_2(u\cdot\nabla b),\partial_2 b)_{H^1}| &= \Big|\int u\cdot\nabla b\cdot\partial_2^2 b\,dx + \int (\nabla u\cdot\nabla b)\cdot\partial_2^2\nabla b\,dx \\ &\quad + \int (u\cdot\nabla)\nabla b\cdot\partial_2^2\nabla b\,dx\Big| \\ &\le \|u\|_{L^\infty}(\|\nabla b\|_{L^2}\|\partial_2^2 b\|_{L^2} + \|\nabla^2 b\|_{L^2}\|\partial_2^2\nabla b\|_{L^2}) \\ &\quad + \|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}\|\partial_2^2\nabla b\|_{L^2} \\ &\le C\|u\|_{H^3}\|\nabla b\|_{H^2}^2 \le C\varepsilon^3, \end{aligned} \tag{101}$$

$$|-(\partial_2(b\cdot\nabla u),\partial_2 b)_{H^1}| = \left|\int b\cdot\nabla u\cdot\partial_2^2 b\,dx + \int(\nabla b\cdot\nabla u)\cdot\partial_2^2\nabla b\,dx + \int(b\cdot\nabla^2 u)\cdot\partial_2^2\nabla b\,dx\right| \leq C\|b\|_{H^3}^2\|\nabla u\|_{H^1} \leq C\varepsilon^3, \tag{102}$$

and

$$|(\partial_2^2 u,\partial_2 b)_{H^1}| \leq \frac{1}{2}(\|\partial_2^2 u\|_{H^1}^2 + \|\partial_2 b\|_{H^1}^2) \leq C\varepsilon^2. \tag{103}$$

As a consequence, we get

$$\left|\frac{d}{dt}\|\partial_2 b(t)\|_{H^1}^2 + 2\gamma\|\partial_2^2 b\|_{H^1}^2\right| \leq C(\varepsilon). \tag{104}$$

The uniform continuity for $\|\partial_2 b\|_{H^1}$ then follows from (104) and the uniform bound of $\|\partial_2^2 b\|_{H^1}^2$. We thus establish the large time behavior for $\|\partial_2 b(t)\|_{H^1}$. This completes the proof of (12) and thus Theorem 1. ■

ACKNOWLEDGMENTS

Lin was partially supported by the National Natural Science Foundation of China (NNSFC) under Grant 11701049 and the Natural Science Foundation of SiChuan Province under Grant 2023NSFSC0056. Wu was partially supported by the National Science Foundation of the United States under grant DMS 2104682 and the AT&T Foundation at Oklahoma State University.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

1. Davidson PA. *An Introduction to Magnetohydrodynamics*. Cambridge University Press; 2001.
2. Laudau L, Lifshitz E. *Electrodynamics of Continuous Media*, 2nd ed. Pergamon; 1984.
3. Priest E, Forbes T. *Magnetic Reconnection, MHD Theory and Applications*. Cambridge University Press; 2000.
4. Wei D, Zhang Z. Global well-posedness for the 2-D MHD equations with magnetic diffusion. *Commun Math Res*. 2020;36:377-389.
5. Cao C, Wu J. Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. *Adv Math*. 2011;226:1803-1822.
6. Cao C, Wu J, Yuan B. The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion. *SIAM J Math Anal*. 2014;46:588-602.
7. Jiu Q, Niu D, Wu J, Xu X, Yu H. The 2D magnetohydrodynamic equations with magnetic diffusion. *Nonlin*. 2015;28:3935-3956.
8. Jiu Q, Zhao J. A remark on global regularity of 2D generalized magnetohydrodynamic equations. *J Math Anal Appl*. 2014;412:478-484.
9. Yamazaki K. On the global regularity of two-dimensional generalized magnetohydrodynamics system. *J Math Anal Appl*. 2014;416:99-111.
10. Yuan B, Zhao J. Global regularity of 2D almost resistive MHD equations. *Nonlin Anal Real World Appl*. 2018;41:53-65.
11. Broadman N, Lin H, Wu J. Stabilization of a background magnetic field on a 2 dimensional magnetohydrodynamic flow. *SIAM J Math Anal*. 2020;52:5001-5035.
12. Majda A, Bertozzi A. *Vorticity and Incompressible Flow*. Cambridge University Press; 2002.

13. Yudovich VI. Non-stationary flows of an ideal incompressible fluid Russian). *Z Vycisl Mat i Mat Fiz*. 1963;3:1032-1066.
14. Tao T. Nonlinear dispersive equations: local and global analysis. In: CBMS Regional Conference Series in Mathematics, Providence, RI: American Mathematical Society, 2006.
15. Alfvén H. Existence of electromagnetic-hydrodynamic waves. *Nature*. 1942;150:405-406.
16. Duvaut G, Lions J. Inéquations en thermoélasticité et magnétohydrodynamique. *Arch Ration Mech Anal*. 1972;46:241-279.
17. Sermange M, Temam R. Some mathematical questions related to the MHD equations. *Commun Pure Appl Math*. 1983;36:635-664.
18. Bardos C, Sulem C, Sulem PL. Longtime dynamics of a conductive fluid in the presence of a strong magnetic field. *Trans Amer Math Soc*. 1988;305:175-191.
19. Cao C, Regmi D, Wu J. The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. *J Differ Equ*. 2013;254:2661-2681.
20. Dai Y, Tang Z, Wu J. A class of global large solutions to the magnetohydrodynamic equations with fractional dissipation. *Z Angew Math Phys*. 2019;70:153.
21. Deng W, Zhang P. Large time behavior of solutions to 3-D MHD system with initial data near equilibrium. *Arch Ration Mech Anal*. 2018;230:1017-1102.
22. Dong B, Jia Y, Li J, Wu J. Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion. *J Math Fluid Mech*. 2018;20:1541-1565.
23. Dong B, Li J, Wu J. Global regularity for the 2D MHD equations with partial hyperresistivity. *Int Math Res Notices*. 2019;14:4261-4280.
24. Du L, Zhou D. Global well-posedness of 2D magnetohydrodynamics flows with partial dissipation and magnetic diffusion. *SIAM J Math Anal*. 2015;47:1562-1587.
25. Feng W, Hafeez F, Wu J. Influence of a background magnetic field on a 2D magnetohydrodynamic flow. *Nonlin*. 2021;34:2527-2562.
26. He L, Xu L, Yu P. On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves. *Ann PDE*. 2018;4:Art.5,105.
27. Hu X. Global existence for two dimensional compressible magnetohydrodynamic flows with zero magnetic diffusivity, 2014, arXiv: 1405.0274v1.
28. Hu X, Lin F. Global existence for two dimensional incompressible magnetohydrodynamic flows with zero magnetic diffusivity, 2014, arXiv: 1405.0082v1.
29. Hu X, Wang D. Low Mach number limit of viscous compressible magnetohydrodynamic flows. *SIAM J Math Anal*. 2009;41:1272-1294.
30. Hu X, Wang D. Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. *Arch Ration Mech Anal*. 2010;197:203-238.
31. Ji R, Wu J. The resistive magnetohydrodynamic equation near an equilibrium. *J Differ Equ*. 2020;268:1854-1871.
32. Jiu Q, Suo X, Wu J, Yu H. Unique weak solutions of the non-resistive magnetohydrodynamic equations with fractional dissipation. *Comm Math Sci*. 2020;18:987-1022
33. Lai S, Wu J, Zhang J. Stabilizing phenomenon for 2D anisotropic magnetohydrodynamic system near a background magnetic field. *SIAM J Math Anal*. 2021;53:6073-6093.
34. Li C, Wu J, Xu X. Smoothing and stabilization effects of magnetic field on electrically conducting fluids. *J Differ Equ*. 2021;276:368-403.
35. Li J, Tan W, Yin Z. Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces. *Adv Math*. 2017;317:786-798.
36. Lin H, Du L. Regularity criteria for incompressible magnetohydrodynamics equations in three dimensions. *Nonlinarity*. 2013;26:219-239.
37. Lin H, Ji R, Wu J, Yan L. Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation. *J Funct Anal*. 2020;279:108519.
38. Lin F, Xu L, Zhang P. Global small solutions to 2-D incompressible MHD system. *J Differ Equ*. 2015;259:5440-5485.
39. Pan R, Zhou Y, Zhu Y. Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes. *Arch Ration Mech Anal*. 2018;227:637-662.

40. Ren X, Wu J, Xiang Z, Zhang Z. Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion. *J Funct Anal*. 2014;267:503-541.
41. Ren X, Xiang Z, Zhang Z. Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain. *Nonlin*. 2016;29:1257-1291.
42. Schonbek M, Schonbek T, Süli E. Large-time behaviour of solutions to the magnetohydrodynamics equations. *Math Ann*. 1996;304:717-756.
43. Tan Z, Wang Y. Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems. *SIAM J Math Anal*. 2018;50:1432-1470.
44. Wan R. On the uniqueness for the 2D MHD equations without magnetic diffusion. *Nonlin Anal Real World Appl*. 2016;30:32-40.
45. Wei D, Zhang Z. Global well-posedness of the MHD equations in a homogeneous magnetic field. *Anal PDE*. 2017;10:1361-1406. https://msp.org/apde/2019/12-2/apde-v12-n2-p.pdf
46. Wu J. The 2D magnetohydrodynamic equations with partial or fractional dissipation. In: *Lectures on the Analysis of Nonlinear Partial Differential Equations*, *Morningside Lectures on Mathematics*, Part 5, MLM5, International Press; 2018:283-332.
47. Wu J, Wu Y. Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion. *Adv Math*. 2017;310:759-888.
48. Wu J, Wu Y, Xu X. Global small solution to the 2D MHD system with a velocity damping term. *SIAM J Math Anal*. 2015;47:2630-2656.
49. Wu J, Zhu Y. Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium. *Adv Math*. 2021;377:107466.
50. Yamazaki K. On the global well-posedness of N-dimensional generalized MHD system in anisotropic spaces. *Adv Differ Equ*. 2014;19:201-224.
51. Yamazaki K. Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation. *Nonlin Anal*. 2014;94:194-205.
52. Yamazaki K. Global regularity of logarithmically supercritical MHD system with zero diffusivity. *Appl Math Lett*. 2014;29:46-51.
53. Yang W, Jiu Q, Wu J. The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation. *J Differ Equ*. 2019;266:630-652.
54. Zhang T. An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system. 2014, arXiv:1404.5681.
55. Zhang T. Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field. *J Differ Equ*. 2016;260:5450-5480.
56. Zhou Y, Zhu Y. Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain. *J Math Phys*. 2018;59:081505.
57. Doering C, Wu J, Zhao K, Zheng X. Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. *Phisica D*. 2018;376/377:144-159.
58. Stein E. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press; 1970.

How to cite this article: Chen T, Lin H, Wu J. The 2D magnetohydrodynamic system with Euler-like velocity equation and partial magnetic diffusion. *Stud Appl Math*. 2023;150:629–649. https://doi.org/10.1111/sapm.12551