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Global Small Solutions to a Special $2\frac{1}{2}$-D Compressible Viscous Non-resistive MHD System

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Abstract

This paper solves the global well-posedness and stability problem on a special $2\frac{1}{2}$-D compressible viscous non-resistive MHD system near a steady-state solution. The steady-state here consists of a positive constant density and a background magnetic field. The global solution is constructed in L^p-based homogeneous Besov spaces, which allow general and highly oscillating initial velocity. The well-posedness problem studied here is extremely challenging due to the lack of the magnetic diffusion and remains open for the corresponding 3D MHD equations. Our approach exploits the enhanced dissipation and stabilizing effect resulting from the background magnetic field, a phenomenon observed in physical experiments. In addition, we obtain the solution's optimal decay rate when the initial data is further assumed to be in a Besov space of negative index.

Keywords Global solutions · Non-resistive compressible MHD · Decay rates

Mathematics Subject Classification 35Q35 · 35A01 · 35A02 · 76W05

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1 Introduction and the Main Results

The small data global well-posedness problem on the three-dimensional (3D) compressible viscous non-resistive magnetohydrodynamic (MHD) equations remains an challenging open problem. Mathematically the concerned MHD equations are given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \mu \Delta \mathbf{v} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{v} + \nabla P = (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \\ \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.1}$$

where ρ denotes the density of the fluid, $\mathbf{v}$ the velocity field, and $\mathbf{B}$ the magnetic field. The parameters μ and λ are shear viscosity and volume viscosity coefficients, respectively, which satisfy the standard strong parabolicity assumption,

$$\mu > 0 \quad \text{and} \quad \nu \overset{\text{def}}{=} \lambda + 2\mu > 0.$$

The pressure $P = A\rho^{\gamma}$ for some $A > 0$ and $\gamma \geq 1$. The compressible MHD equations model the motion of electrically conducting fluids in the presence of a magnetic field. The compressible MHD equations can be derived from the isentropic Navier–Stokes–Maxwell system by taking the zero dielectric constant limit. When the effect of the magnetic field can be neglected or $\mathbf{B} = 0$, (1.1) reduces to the isentropic compressible Navier–Stokes equations.

The goal of this paper is to solve the small data global well-posedness problem on a very special two-and-half-dimensional ($2\frac{1}{2}$-D) compressible viscous non-resistive MHD equations (to be specified later). In addition, we are also interested in the precise large-time behavior of the solutions.

Due to its wide physical applications and mathematical challenges, the compressible MHD equations have attracted the interests of many physicists and mathematicians (see, e.g., Bian and Guo 2016; Davidson 2017; Dou et al. 2013; Ducomet and Feireisl

2006; Fan and Yu 2009; Feireisl et al. 2014; Hao 2011; Hoff and Tsyganov 2005; Wu and Wu 2017; Zhu 2015; Zhong 2020 and the references therein). We briefly recall some results concerning the multi-dimensional barotropic compressible MHD equations, which are closely related to our investigation here. Ducomet and Feireisl (2006) considered the heat-conducting fluids together with the influence of radiation and obtained the global existence of weak solutions with finite energy initial data. Hu and Wang (2010) proved the global existence of weak solutions to the 3D isentropic compressible MHD system via the Lions-Feireisl theory, see (Lions 1998) and (Feireisl 2004). We remark that there are essential differences between the vacuum case and the non-vacuum case. The global weak solution in the case of vacuum was obtained in the work of Li et al. (2013). The local well-posedness in the framework of critical Besov spaces was shown by Bian and Guo (2016) when there is full dissipation and no vacuum. In the case of vacuum and no magnetic diffusion, (Li et al. 2011) proved the local existence and uniqueness of strong solutions. The small data global well-posedness problem is extremely difficult when there is no magnetic diffusion. There are some satisfactory results in the simplified 1D geometry. Jiang and Zhang (2017) proved the existence and uniqueness of global strong solution to the isentropic case with large initial data. We refer to Li (2018), Li and Jiang (2019) for more results in 1D concerning isentropic and heat-conductive non-resistive MHD system with large initial data. Wu and Wu (2017) presented a systematic approach to the small data global well-posedness and stability problem on the 2D compressible non-resistive MHD equations if the initial data are close to an equilibrium state, especially with a background magnetic field. It appears difficult to extend the approach of Wu and Wu (2017) to $\mathbb{R}^3$. There are some differences between 2D case and 3D case. For 2D case, when applying ∇ on equations, there will appear at least one good part in nonlinear terms. For example, $\partial_1 \mathbf{v} \cdot \nabla \mathbf{B} = \partial_1 \mathbf{v}_1 \partial_1 \mathbf{B} + \partial_1 \mathbf{v}_2 \partial_2 \mathbf{B}$ and $\partial_2 \mathbf{v} \cdot \nabla \mathbf{B}$ (coming from $\nabla \mathbf{v} \cdot \nabla \mathbf{B}$) always contain a strong dissipative part. However, this will not hold for 3D case. Tan and Wang (2018) obtained the global existence of smooth solutions to the 3D compressible barotropic viscous non-resistive MHD system in the horizontally infinite flat layer $\Omega = \mathbb{R}^2 \times (0, 1)$. Initial- and boundary-value problems under some additional compatibility conditions for the 3D compressible MHD equations were examined by Fan and Yu (2009) and local solutions were obtained even when there is a vacuum. Zhu (2015) extended the result obtained in Li et al. (2011) to the case of allowing non-negativity of the initial density. We mention that there are many interesting results on the zero Mach limit results on the incompressible MHD equations (see, e.g., Dou et al. 2013; Hu and Wang 2009; Feireisl et al. 2014; Jiang et al. 2010a; Li 2012; Li et al. 2017).

If we neglect the effect of the magnetic field, the system (1.1) reduces to the compressible Navier–Stokes equations, which have also been studied by many researchers, see (Danchin 2000, 2014; Danchin and He 2016; Danchin and Xu 2017; Huang et al. 2012; Jiu et al. 2018; Lei and Xin 2019; Li and Xin 2019; Xin and Yan 2013; Xin and Zhu 2021; Zhai and Chen 2020) and the references therein.

Although the small data global well-posedness on the 2D compressible MHD equations without magnetic diffusion has been successfully settled, this same problem on the 3D counterpart appears to be inaccessible at this moment. This paper focuses on a very special $2\frac{1}{2}$-D compressible MHD system. The motion of fluids takes place in the plane $\mathbb{R}^2$ while the magnetic field acts on fluids only in the vertical direction, namely

$$\mathbf{v} = (\mathbf{v}^1(t, x_1, x_2), \mathbf{v}^2(t, x_1, x_2), 0) \stackrel{\text{def}}{=} (\mathbf{u}, 0),$$
$$\rho \stackrel{\text{def}}{=} \rho(t, x_1, x_2), \quad \mathbf{B} \stackrel{\text{def}}{=} (0, 0, m(t, x_1, x_2)).$$

Then (1.1) is reduced to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P + \frac{1}{2} \nabla m^2 = 0, \\ \partial_t m + \operatorname{div}(m \mathbf{u}) = 0. \end{cases} \tag{1.2}$$

Clearly $(\rho^{(0)}, \mathbf{u}^{(0)}, m^{(0)})$ with

$$\rho^{(0)} = 1, \quad \mathbf{u}^{(0)} = 0, \quad m^{(0)} = 1$$

solves (1.2). We intend to understand the well-posedness and stability problem on the system governing the perturbation $(a, \mathbf{u}, b)$, where

$$a = \rho - 1, \quad b = m - 1.$$

It is easy to check that $(a, \mathbf{u}, b)$ satisfies

$$\begin{cases} \partial_t a + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla a + a \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P(1 + a) + \frac{1}{2} \nabla (b + 1)^2 = \mathbf{M}(a, \mathbf{u}, b), \\ \partial_t b + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla b + b \operatorname{div} \mathbf{u} = 0, \\ (a, \mathbf{u}, b)|_{t=0} = (a_0, \mathbf{u}_0, b_0), \end{cases} \tag{1.3}$$

with

$$\mathbf{M}(a, \mathbf{u}, b) \stackrel{\text{def}}{=} \frac{a}{1 + a} (\nabla P(1 + a) + \frac{1}{2} \nabla (b + 1)^2 - (\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u})). \tag{1.4}$$

As the first step of our main results, we provide a local well-posedness result in the Besov space.

Proposition 1.1 (Local well-posedness) *Let $1 < p < 4$. Assume $\mathbf{u}_0 \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$, $(a_0, b_0) \in \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$ with $1 + a_0$ bounded away from zero. Then there exists a positive time T such that the system (1.3) has a unique solution $(a, \mathbf{u}, b)$ satisfying*

$$(a, b) \in C([0, T]; \dot{B}_{p,1}^{\frac{2}{p}}), \quad \mathbf{u} \in C([0, T]; \dot{B}_{p,1}^{\frac{2}{p}-1}) \cap L^1([0, T]; \dot{B}_{p,1}^{\frac{2}{p}+1}).$$

Before stating our main results, we introduce some notation. Let $\mathscr{S}(\mathbb{R}^2)$ be the Schwartz space on $\mathbb{R}^2$ and $\mathscr{S}'(\mathbb{R}^2)$ be its dual space. For any $z \in \mathscr{S}'(\mathbb{R}^2)$, the lower- and higher-frequency parts are expressed as[1]

$$z^\ell \stackrel{\text{def}}{=} \sum_{j\le j_0} \dot\Delta_j z \quad \text{and} \quad z^h \stackrel{\text{def}}{=} \sum_{j\ge j_0-1} \dot\Delta_j z$$

for some fixed integer j_0 (the value of j_0 is fixed in the proofs of the main theorems). The corresponding truncated semi-norms are defined as follows:

$$\|z\|^\ell_{\dot B^s_{p,1}} \stackrel{\text{def}}{=} \|z^\ell\|_{\dot B^s_{p,1}} \quad \text{and} \quad \|z\|^h_{\dot B^s_{p,1}} \stackrel{\text{def}}{=} \|z^h\|_{\dot B^s_{p,1}}.$$

Let $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ be the projection onto the divergence-free vector fields and $\mathbb{Q} = I - \mathbb{P} = \nabla\Delta^{-1}\nabla\cdot$.

The small data global well-posedness and stability result on (1.3) is stated in the following theorem.

Theorem 1.2 (Global well-posedness) *Let* $2 \le p < 4$. *For any* $(a_0^\ell, \mathbb{Q}\mathbf{u}_0^\ell, b_0^\ell) \in \dot B^0_{2,1}(\mathbb{R}^2)$, $(a_0^h, b_0^h) \in \dot B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ *and* $(\mathbb{P}\mathbf{u}_0, \mathbb{Q}\mathbf{u}_0^h) \in \dot B^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$, *there exists a positive constant* c_0 *such that, if*

$$\|(a_0^\ell, \mathbb{Q}\mathbf{u}_0^\ell, b_0^\ell)\|_{\dot B^0_{2,1}} + \|(a_0^h, b_0^h)\|_{\dot B^{\frac{2}{p}}_{p,1}} + \|(\mathbb{P}\mathbf{u}_0, \mathbb{Q}\mathbf{u}_0^h)\|_{\dot B^{\frac{2}{p}-1}_{p,1}} \le c_0, \tag{1.5}$$

then the system (1.3) *has a unique global solution* $(a, \mathbf{u}, b)$ *so that*

$$(a^\ell, b^\ell) \in C_b(\mathbb{R}^+; \dot B^0_{2,1}), \quad (a^h, b^h) \in C_b(\mathbb{R}^+; \dot B^{\frac{2}{p}}_{p,1}),$$
$$\mathbb{Q}\mathbf{u}^\ell \in C_b(\mathbb{R}^+; \dot B^0_{2,1} \cap L^1(\mathbb{R}^+; \dot B^2_{2,1}), \quad (\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h) \in C_b(\mathbb{R}^+; \dot B^{\frac{2}{p}-1}_{p,1} \cap L^1(\mathbb{R}^+; \dot B^{\frac{2}{p}+1}_{p,1}).$$

Moreover, there exists some constant C *such that*

$$\mathscr{X}(t) \le Cc_0, \tag{1.6}$$

where

$$\mathscr{X}(t) \stackrel{\text{def}}{=} \|(a^\ell, \mathbb{Q}\mathbf{u}^\ell, b^\ell)\|_{\widetilde L^\infty_t(\dot B^0_{2,1})} + \|(a^h, b^h)\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}}_{p,1})} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}-1}_{p,1})}$$
$$+ \|(\varphi^\ell, \mathbb{Q}\mathbf{u}^\ell)\|_{L^1_t(\dot B^2_{2,1})} + \|\varphi^h\|_{L^1_t(\dot B^{\frac{2}{p}}_{p,1})} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot B^{\frac{2}{p}+1}_{p,1})}.$$

One refers to (4.1) *for the definition of* φ.

[1] Note that for technical reasons, we need a small overlap between low and high frequency.

1.1 Strategy of the Proof of Theorem

1.2

Let us point out new ingredients in the proof of Theorem 1.2. For usual compressible Navier–Stokes equations (see for example (Danchin 2000; Danchin and He 2016)), the major difficulty stems from the convection term in the density equation, as it may cause a loss of one derivative of the density. To overcome it, previous proofs heavily relied on a paralinearized version combined with a Lagrangian change of variables. For the compressible viscous non-resistive MHD system (1.3), the situation becomes more complicated. There are absence of dissipation in the density equation and the magnetic field equation, we cannot get any smoothing effect of the density and the magnetic field. This bring us big difficulty to construct the global solutions of the system. The new ingredient in the present paper lies in the introduction of un unknown good function φ (see (4.1)), which enables us to capture the dissipation arising from combination of density and the magnetic field. Finally, we complete the proof of Theorem 1.2 by a continuous argument.

It is natural and physically important to study the large-time behavior of the global solution obtained in 1.2. The large-time behavior has always been a prominent topic on the fluid equations. Important results have been established for the compressible Navier–Stokes equations (see, e.g., Danchin and Xu 2017; Xin and Xu 2021; Zhai and Chen 2020) and the compressible MHD equations (see, e.g., Hu and Wang 2010).

What is special here is that the system concerned here is partially dissipated with no damping or dissipation in the equations of ρ and b. We show that, when the low modes of the initial data are in a Besov space with suitable negative index, the Sobolev norm of the solution is shown to decay at an optimal rate. The proof relies on the enhanced dissipation resulting from the interaction between the velocity and the magnetic field.

Theorem 1.3 (Optimal decay) *Let* $\Lambda^s z \overset{\text{def}}{=} \mathscr{F}^{-1}(|\xi|^s \mathscr{F} z)(s \in \mathbb{R})$ *and* $(a, \mathbf{u}, b)$ *be the global small solutions addressed by Theorem 1.2 with* $p = 2$*. For any* $0 < \sigma \leq 1$*, if additionally the initial data satisfying* $(a_0^\ell, \mathbf{u}_0^\ell, b_0^\ell) \in \dot{B}^{-\sigma}_{2,\infty}(\mathbb{R}^2)$*, then we have the following time-decay rate*

$$\|\Lambda^{\gamma_1}(\varphi, \mathbf{u})\|_{L^2} \leq C(1+t)^{-\frac{\gamma_1+\sigma}{2}}, \qquad \forall \gamma_1 \in (-\sigma, 0] \tag{1.7}$$

with φ *is defined in* (4.1).

Remark 1.4 The above decay rate (1.7) coincides with the heat flows, thus it is optimal in some sense.

Finally, we mention the small data global well-posedness result for a closely related system of inhomogeneous incompressible MHD equations. The general inhomogeneous incompressible MHD equations are of the form

$$
\begin{cases}
\partial_t \rho + \mathrm{div}\,(\rho \mathbf{v}) = 0, \\
\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \mu \Delta \mathbf{v} + \nabla P = (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \\
\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{B} = 0, \\
(\rho, \mathbf{v}, \mathbf{B})|_{t=0} = (\rho_0, \mathbf{v}_0, \mathbf{B}_0).
\end{cases}
\tag{1.8}
$$

If we set

$$
\mathbf{v} = (\mathbf{v}^1(t, x_1, x_2), \mathbf{v}^2(t, x_1, x_2), 0) \stackrel{\text{def}}{=} (\mathbf{u}, 0),
$$
$$
\rho \stackrel{\text{def}}{=} \rho(t, x_1, x_2), \quad \mathbf{B} \stackrel{\text{def}}{=} (0, 0, b(t, x_1, x_2)),
$$

then (1.8) is reduced to

$$
\begin{cases}
\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0, \\
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla P + \dfrac{1}{2} \nabla b^2 = 0, \\
\partial_t b + \mathrm{div}\,(b \mathbf{u}) = 0, \\
\mathrm{div}\,\mathbf{u} = 0, \\
(\rho, \mathbf{u}, b)|_{t=0} = (\rho_0, \mathbf{u}_0, b_0).
\end{cases}
\tag{1.9}
$$

Different from the compressible MHD equations, the combination $\Pi := P + \frac{1}{2} b^2$ can be regarded as new pressure and the new system (1.9) is decoupled into equations of $(\rho, \mathbf{u})$ and the equation of b. We can solve the equations of $(\rho, \mathbf{u})$ first and then get the solution of b through the third equation of (1.9). Now we write $\rho = 1 + a$, inspired by Abidi and Gui (2021), Xu et al. (2016) and the previous well-posedness result on the compressible MHD equations, we obtain the following global well-posedness result on (1.9). We shall not provide a detailed proof for this result.

Theorem 1.5 *Let $p \in (1, 4)$, $(a_0, \mathbf{u}_0, b_0) \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ with $\mathrm{div}\,\mathbf{u}_0 = 0$ and $1 + a_0$ bounded away from zero. Then (1.9) has a unique global solution $(a, \mathbf{u}, \nabla \Pi, b)$ such that for any $t > 0$,*

$$
\begin{aligned}
(a, b) \in & C(\mathbb{R}^+; \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap \widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)), \ \nabla \Pi \in L^1_t(\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)), \\
\mathbf{u} \in & C(\mathbb{R}^+; ; \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)) \cap \widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)) \cap L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1}(\mathbb{R}^2)).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
& \|(a, b)\|_{\widetilde{L}^\infty_t(B^{\frac{2}{p}}_{p,1})} + \|\mathbf{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} + \|\mathbf{u}\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \|\nabla \Pi\|_{L^1_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} \\
& \le C \exp\left(C \exp\left(C t^{\frac{1}{2}}\right)\right)
\end{aligned}
$$

for some time-independent constant C.

The rest of this paper is arranged as follows. In the second section, we recall some basic facts about Littlewood-Paley theory. In the third section, we use the fixed point theorem to outline the proof of Proposition 1.1. In the forth section, we use three subsections to prove Theorem 1.2. In the first subsection, we exploit the special structure of (1.3) to capture the dissipation arising from combination of density and the magnetic field at low-frequency part and in the second subsection, we introduce a so-called effective velocity to capture the dissipation arising from combination of density and the magnetic field at high-frequency part, respectively. In the last subsection, we use the continuity argument to close the energy estimates and thus complete the proof of Theorem 1.2. We shall prove the Theorem 1.3 in Section 5 Inspired by the papers (Xin and Xu 2021), our main task is to establish a Lyapunov-type inequality in time for energy norms (see (5.34)) by using the pure energy argument (independent of spectral analysis).

Let us introduce some notations. For two operators A and B, we denote $[A, B] = AB - BA$, the commutator between A and B. The letter C stands for a generic constant whose meaning is clear from the context. We denote $\langle a, b\rangle$ the $L^2(\mathbb{R}^2)$ inner product of a and b and write $a \lesssim b$ instead of $a \le Cb$. Given a Banach space X, we shall denote $\|(a, b)\|_X = \|a\|_X + \|b\|_X$.

2 Preliminaries

This section reviews Besov spaces and related facts to be used in the subsequent sections. We start with the Littlewood-Paley decomposition. To define it, we fix a smooth radial non-increasing function χ supported in the ball $B(0, \frac{4}{3})$ of $\mathbb{R}^2$, and with value 1 on $B(0, \frac{3}{4})$ such that, for $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$,

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\cdot) = 1 \ \text{ in } \ \mathbb{R}^2 \setminus \{0\} \quad \text{and} \quad \operatorname{Supp}\varphi \subset \left\{\xi \in \mathbb{R}^2 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined on tempered distributions by

$$\dot{\Delta}_j u \stackrel{\text{def}}{=} \varphi(2^{-j}D)u \stackrel{\text{def}}{=} \mathscr{F}^{-1}(\varphi(2^{-j}\cdot)\mathscr{F}u).$$

For any homogeneous function A of order 0 and smooth outside 0, we have

$$\forall p \in [1, \infty], \qquad \|\dot{\Delta}_j(A(D)u)\|_{L^p} \le C\|\dot{\Delta}_j u\|_{L^p}.$$

Definition 2.1 Let p, r be in $[1, +\infty]$ and s in $\mathbb{R}$, $u \in \mathscr{S}'(\mathbb{R}^2)$. We define the Besov norm by

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left\|\left(2^{js}\|\dot{\Delta}_j u\|_{L^p}\right)_j\right\|_{\ell^r(\mathbb{Z})}.$$

We then define the homogeneous Besov spaces by $\dot{B}^s_{p,r} \stackrel{\text{def}}{=} \Big\{u \in \mathscr{S}'_h(\mathbb{R}^2),\ \|u\|_{\dot{B}^s_{p,r}} < \infty\Big\}$, where $u \in \mathscr{S}'_h(\mathbb{R}^2)$ means that $u \in \mathscr{S}'(\mathbb{R}^2)$ and $\lim_{j\to-\infty} \|\dot{S}_j u\|_{L^\infty} = 0$ (see Definition 1.26 of Bahouri et al. (2011)).

When employing parabolic estimates in Besov spaces, it is somehow natural to take the time-Lebesgue norm before performing the summation for computing the Besov norm. So we next introduce the following Besov-Chemin-Lerner space $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ (see Bahouri et al. 2011):

$$\widetilde{L}^q_T(\dot{B}^s_{p,r}) = \Big\{u(t,x) \in (0,+\infty) \times \mathscr{S}'_h(\mathbb{R}^2) : \|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} < +\infty\Big\},$$

where

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \big\|2^{ks}\|\dot{\Delta}_k u(t)\|_{L^q(0,T;L^p)}\big\|_{\ell^r}.$$

The index T will be omitted if $T = +\infty$, and we shall denote by $\widetilde{\mathscr{C}_b}([0,T]; \dot{B}^s_{p,r})$ the subset of functions of $\widetilde{L}^\infty_T(\dot{B}^s_{p,r})$ which are also continuous from $[0,T]$ to $\dot{B}^s_{p,r}$.

By the Minkowski inequality, we have the following inclusions between the Chemin-Lerner space $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$ and the Bochner space $L^\lambda_T(\dot{B}^s_{p,r})$:

$$\|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \le \|u\|_{L^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda \le r, \qquad \|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \ge \|u\|_{L^\lambda_T(\dot{B}^s_{p,r})}, \quad \text{if } \lambda \ge r.$$

The following Bernstein's lemma will be repeatedly used throughout this paper.

Lemma 2.2 *Let $\mathscr{B}$ be a ball and $\mathscr{C}$ a ring of $\mathbb{R}^2$. A constant C exists so that for any positive real number λ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (p,q) with $1 \le p \le q \le \infty$, there hold*

$$\begin{aligned}
\operatorname{Supp}\hat{u} \subset \lambda\mathscr{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \le C^{k+1}\lambda^{k+2(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},\\
\operatorname{Supp}\hat{u} \subset \lambda\mathscr{C} &\Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \le \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \le C^{k+1}\lambda^k\|u\|_{L^p},\\
\operatorname{Supp}\hat{u} \subset \lambda\mathscr{C} &\Rightarrow \|\sigma(D)u\|_{L^q} \le C_{\sigma,m}\lambda^{m+2(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}.
\end{aligned}$$

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Here, we recall the decomposition in the homogeneous context:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v) = \dot{T}_u v + \dot{T}'_v u, \tag{2.1}$$

where

$$\dot{T}_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v, \qquad \dot{R}(u,v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u\widetilde{\dot{\Delta}}_j v,$$

and

$$\widetilde{\dot{\Delta}}_j v \overset{\text{def}}{=} \sum_{|j-j'|\le 1} \dot{\Delta}_{j'} v, \qquad \dot{T}'_v u \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j+2} v \dot{\Delta}_j u.$$

The paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties.

Lemma 2.3 *Let $(s, r) \in \mathbb{R} \times [1, \infty]$ and $1 \le p, p_1, p_2 \le \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

- *We have:*

$$\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^{p_1}} \|v\|_{\dot{B}^s_{p_2,r}} \quad \text{and} \quad \|\dot{T}_u v\|_{\dot{B}^{s+t}_{p,r}} \lesssim \|u\|_{\dot{B}^t_{p_1,\infty}} \|v\|_{\dot{B}^s_{p_2,r}}, \quad \text{if } t < 0.$$

- *If $s_1 + s_2 > 0$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \le 1$ then*

$$\|\dot{R}(u, v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

- *If $s_1 + s_2 = 0$ and $\frac{1}{r_1} + \frac{1}{r_2} \ge 1$ then*

$$\|\dot{R}(u, v)\|_{\dot{B}^0_{p,\infty}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}. \tag{2.2}$$

From Lemma 2.3, we may deduce the following several nonlinear estimates in Besov spaces

Lemma 2.4 (Bahouri et al. 2011) *Let $s_1 \le \frac{2}{p}$, $s_2 < \frac{2}{p}$, $s_1 + s_2 \ge 2\max(0, \frac{2}{p} - 1)$, and $1 \le p \le \infty$. Assume that $u \in \dot{B}^{s_1}_{p,1}(\mathbb{R}^2)$ and $v \in \dot{B}^{s_2}_{p,\infty}(\mathbb{R}^2)$. Then there holds*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{2}{p}}_{p,\infty}} \le C\|u\|_{\dot{B}^{s_1}_{p,1}} \|v\|_{\dot{B}^{s_2}_{p,\infty}}.$$

Lemma 2.5 (Xin and Xu 2021, Proposition A.1) *Let $1 \le p, q \le \infty$, $s_1 \le \frac{2}{q}$, $s_2 \le 2\min\{\frac{1}{p}, \frac{1}{q}\}$ and $s_1+s_2 > 2\max\{0, \frac{1}{p}+\frac{1}{q}-1\}$. For any $(u, v) \in \dot{B}^{s_1}_{q,1}(\mathbb{R}^2)\times\dot{B}^{s_2}_{p,1}(\mathbb{R}^2)$, we have*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{2}{q}}_{p,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{q,1}} \|v\|_{\dot{B}^{s_2}_{p,1}}.$$

Lemma 2.6 *Let $2 \le p < 4$. For any $u \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, $v^\ell \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ and $v^h \in \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$, we have*

$$\|(uv)^\ell\|_{\dot{B}^0_{2,1}} \lesssim (\|v^\ell\|_{\dot{B}^0_{2,1}} + \|v^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}})\|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}. \tag{2.3}$$

Proof We first use Bony's decomposition to write

$$\dot{S}_{j_0+1}(uv) = \dot{T}_u \dot{S}_{j_0+1} v + \dot{S}_{j_0+1}\big(\dot{T}_v u + \dot{R}(v,u)\big) + [\dot{S}_{j_0+1}, \dot{T}_u]v. \tag{2.4}$$

Applying Lemma 2.3, we have

$$\|\dot{T}_u \dot{S}_{j_0+1} v\|_{\dot{B}^0_{2,1}} \lesssim \|u\|_{L^\infty} \|v^\ell\|_{\dot{B}^0_{2,1}} \lesssim \|v^\ell\|_{\dot{B}^0_{2,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}, \tag{2.5}$$

and, for $\frac{1}{p*} = \frac{1}{2} - \frac{1}{p}$,

$$\|\dot{S}_{j_0+1} \dot{T}_v u\|_{\dot{B}^0_{2,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p^*}-1}_{p^*,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}. \tag{2.6}$$

For the reminder term $\|\dot{S}_{j_0+1} \dot{R}(v,u)\|_{\dot{B}^0_{2,1}}$, we cannot use Lemma 2.3 directly, however, in view of the fact that $1 \le \frac{p}{2} < 2$ and $\frac{4}{p} - 1 > 0$, there holds

$$\|\dot{S}_{j_0+1} \dot{R}(v,u)\|_{\dot{B}^0_{2,1}} \lesssim \|\dot{R}(v,u)\|_{\dot{B}^{\frac{4}{p}-1}_{p/2,1}} \tag{2.7}$$

from which and Lemma 2.3, we get

$$\|\dot{S}_{j_0+1} \dot{R}(v,u)\|_{\dot{B}^0_{2,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}. \tag{2.8}$$

By Lemma 6.1 in Danchin and He (2016), the term with the commutator can be bounded

$$\|[\dot{S}_{j_0+1}, \dot{T}_u]v\|_{\dot{B}^0_{2,1}} \lesssim \|\nabla u\|_{\dot{B}^{\frac{2}{p^*}-1}_{p^*,1}} \|v\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}. \tag{2.9}$$

Thus, the combination of (2.4)–(2.9) shows the validity of (2.3). □

We also need the following classical commutator's estimate.

Lemma 2.7 (Bahouri et al. 2011, Lemma 2.100) *Let* $1 \le p \le \infty$, $-2\min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\} < s \le \frac{2}{p}$. *For any* $v \in \dot{B}^s_{p,1}(\mathbb{R}^2)$ *and* $\nabla u \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, *there holds*

$$\big\|[\dot{\Delta}_j, u\cdot\nabla]v\big\|_{L^p} \lesssim d_j 2^{-js} \|\nabla u\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \|v\|_{\dot{B}^s_{p,1}}$$

where $(d_j)_{\ell^1} = 1$.

Finally, we recall a composition result and the parabolic regularity estimate for the heat equation to end this section.

Lemma 2.8 (Bahouri et al. 2011) *Let G with $G(0)=0$ be a smooth function defined on an open interval I of $\mathbb{R}$ containing 0. Then the following estimates*

$$\|G(f)\|_{\dot{B}^s_{p,1}} \lesssim \|f\|_{\dot{B}^s_{p,1}} \quad \textit{and} \quad \|G(f)\|_{\widetilde{L}^q_T(\dot{B}^s_{p,1})} \lesssim \|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,1})}$$

hold true for $s>0$, $1\le p,\ q\le\infty$ and f valued in a bounded interval $J\subset I$.

Lemma 2.9 (Bahouri et al. 2011) *Let $\sigma\in\mathbb{R}$, $T>0$, $1\le p,r\le\infty$ and $1\le q_2\le q_1\le\infty$. Let u satisfy the heat equation*

$$\partial_t u-\Delta u=f,\quad u|_{t=0}=u_0.$$

Then there holds the following a priori estimate

$$\|u\|_{\widetilde{L}^{q_1}_T(\dot{B}^{\sigma+\frac{2}{q_1}}_{p,r})} \lesssim \|u_0\|_{\dot{B}^{\sigma}_{p,r}}+\|f\|_{\widetilde{L}^{q_2}_T(\dot{B}^{\sigma-2+\frac{2}{q_2}}_{p,r})}.$$

3 The Proof of Proposition 1.1

We prove Proposition 1.1 by a fixed point theorem under the Lagrangian coordinates. We follow the paper (Danchin 2014) closely and only give the sketch of the proof.

Step 1. First, we convert (1.2) into is Lagrangian formulation. For this, we need to introduce some notations. For a vector $\mathbf{w}=\mathbf{w}(x)=(w_1,w_2)$, $\nabla_x\mathbf{w}$ denotes the matrix $(\partial_{x_i}w_j)_{ij}$ and $D_x\mathbf{w}=(\nabla_x\mathbf{w})^\intercal$ (i.e., the transpose of $\nabla_x\mathbf{w}$). We may also frequently write $\nabla\mathbf{w}$ and $D\mathbf{w}$ when it is clear which space variable $\mathbf{w}$ depends on.

If $\mathbf{u}=\mathbf{u}(t,x)$ is a C^1 vector field, it uniquely determines a trajectory $X(t,\cdot)$, defined by the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{dt}X(t,y)=\mathbf{u}(t,X(t,y)),\\ X(0,y)=y. \end{cases} \tag{3.1}$$

Moreover, $X(t,\cdot)$ is a C^1-diffeomorphism over $\mathbb{R}^2$ for every $t\ge 0$.

We are about to reformulate (1.2) using the following unknowns in Lagrangian coordinates:

$$\bar\rho(t,y)\stackrel{\text{def}}{=}\rho\big(t,X(t,y)\big),\quad \bar m(t,y)\stackrel{\text{def}}{=}m\big(t,X(t,y)\big),\quad \text{and}\quad \bar{\mathbf{u}}(t,y)\stackrel{\text{def}}{=}\mathbf{u}\big(t,X(t,y)\big). \tag{3.2}$$

We may keep in mind that now X only depends on $\bar{\mathbf{u}}$ since

$$X(t,y)=y+\int_0^t\bar{\mathbf{u}}(\tau,y)\,\mathrm{d}\tau. \tag{3.3}$$

Next, let us introduce $J_{\bar{\mathbf{u}}}(t,y) = \det DX(t,y)$. Then $(1.2)_1$ and $(1.2)_3$ imply, respectively,

$$J_{\bar{\mathbf{u}}}\bar{\rho} \equiv \rho_0, \quad \text{and} \quad J_{\bar{\mathbf{u}}}\bar{m} \equiv m_0. \tag{3.4}$$

To reformulate $(1.2)_2$, we further introduce $A_{\bar{\mathbf{u}}}(t,y) = \big(DX(t,y)\big)^{-1}$ and $\mathscr{A}_{\bar{\mathbf{u}}}(t,y) = \operatorname{adj} DX(t,y)$ (the adjugate of DX, i.e., $\mathscr{A}_{\bar{\mathbf{u}}} = J_{\bar{\mathbf{u}}}A_{\bar{\mathbf{u}}}$). As in Danchin (2014), evaluating $(1.2)_2$ at $(t, X(t,y))$, multiplying the resulting equation by $J_{\bar{\mathbf{u}}}$, and using (3.4), we have

$$\begin{cases} \rho_0\partial_t\bar{\mathbf{u}} - \mu\operatorname{div}(\mathscr{A}_{\bar{\mathbf{u}}}A_{\bar{\mathbf{u}}}^{\mathsf{T}}\nabla\bar{\mathbf{u}}) - (\mu+\lambda)\mathscr{A}_{\bar{\mathbf{u}}}^{\mathsf{T}}\nabla\operatorname{Tr}(A_{\bar{\mathbf{u}}}D\bar{\mathbf{u}}) \\ \quad + \mathscr{A}_{\bar{\mathbf{u}}}^{\mathsf{T}}\nabla(P(J_{\bar{\mathbf{u}}}^{-1}\rho_0) + \frac{1}{2}(J_{\bar{\mathbf{u}}}^{-1}m_0)^2) = 0, \\ \bar{\mathbf{u}}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{3.5}$$

Here, Tr denotes the trace of square matrices.

Step 2. Linearized system. Note that (3.5) is already a determined system with $\bar{\mathbf{u}}$ the only unknown. Since it is fully nonlinear, we need to reformulate it as

$$\begin{cases} \rho_0\partial_t\bar{\mathbf{u}} - \mu\Delta\bar{\mathbf{u}} - (\mu+\lambda)\nabla\operatorname{div}\bar{\mathbf{u}} = f(\bar{\mathbf{u}}), \\ \bar{\mathbf{u}}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} f(\bar{\mathbf{u}}) =& \mu\operatorname{div}((\mathscr{A}_{\bar{\mathbf{u}}}A_{\bar{\mathbf{u}}}^{\mathsf{T}} - I)\nabla\bar{\mathbf{u}}) + (\mu+\lambda)(\mathscr{A}_{\bar{\mathbf{u}}}^{\mathsf{T}} - I)\nabla\operatorname{Tr}(A_{\bar{\mathbf{u}}}D\bar{\mathbf{u}}) \\ &+ (\mu+\lambda)\nabla\operatorname{Tr}((A_{\bar{\mathbf{u}}} - I)D\bar{\mathbf{u}}) - \mathscr{A}_{\bar{\mathbf{u}}}^{\mathsf{T}}\nabla(P(J_{\bar{\mathbf{u}}}^{-1}\rho_0) + \frac{1}{2}(J_{\bar{\mathbf{u}}}^{-1}m_0)^2) \end{aligned}$$

and I is the identity matrix.

We need the following well-posedness result for the linearized system.

Theorem 3.1 (See Xu 2022) *Let $1 < p < 4$. Assume $\mathbf{u}_0 \in \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$, $\rho_0 - 1 \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, and $\inf\rho_0 > 0$. If $f \in L^1([0,T];\dot{B}^{\frac{2}{p}-1}_{p,1})$ for some positive time T, then the system*

$$\begin{cases} \rho_0\partial_t\bar{\mathbf{u}} - \mu\Delta\bar{\mathbf{u}} - (\mu+\lambda)\nabla\operatorname{div}\bar{\mathbf{u}} = f, \\ \bar{\mathbf{u}}|_{t=0} = \mathbf{u}_0 \end{cases}$$

has a unique solution $\bar{\mathbf{u}}$ in the class $C([0,T];\dot{B}^{\frac{2}{p}-1}_{p,1}) \cap L^1([0,T];\dot{B}^{\frac{2}{p}+1}_{p,1})$.

Moreover, we have the global estimate

$$\|\bar{\mathbf{u}}\|_{L^\infty_T(\dot{B}^{\frac{2}{p}-1}_{p,1})} + \|(\partial_t\bar{\mathbf{u}}, \Delta\bar{\mathbf{u}})\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})} \le C\|\mathbf{u}_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + C\|f\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})},$$

where C depends on p, $\inf \rho_0, \mu, \lambda, \|\rho_0 - 1\|_{\dot{B}^{\frac{2}{p}}_{p,1}}$ *but T*.

In fact, the author in Xu (2022) did not discuss the case $p = 2$. However, our regularity of the initial density ρ_0 is much higher than that in Xu (2022). Then one can follow the argument in Xu (2022) to show Theorem 3.1 for $p = 2$. On the other hand, we can also use the linear theory established in Danchin (2014) to show the local well-posedness of (3.5). But in Danchin (2014), the constant C in the linear estimate depends on T.

Step 3. Fixed point argument. We shall perform the fixed point theorem in the Banach space $E_p(T)$ defined as

$$E_p(T) \stackrel{\text{def}}{=} \left\{ \bar{\mathbf{u}} \in C_b([0,T]; \dot{B}^{\frac{2}{p}-1}_{p,1}) | \partial_t \bar{\mathbf{u}} \in L^1([0,T]; \dot{B}^{\frac{2}{p}-1}_{p,1}), \bar{\mathbf{u}} \in L^1([0,T]; \dot{B}^{\frac{2}{p}+1}_{p,1}) \right\}$$

endowed with the norm

$$\|\bar{\mathbf{u}}\|_{E_p} \stackrel{\text{def}}{=} \|\bar{\mathbf{u}}\|_{L^\infty_T(\dot{B}^{\frac{2}{p}-1}_{p,1})} + \|(\partial_t \bar{\mathbf{u}}, \Delta \bar{\mathbf{u}})\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})}.$$

We need the nonlinear estimates for $f(\bar{\mathbf{u}})$ when $\bar{\mathbf{u}} \in E_p(T)$ satisfying $\|\Delta \bar{\mathbf{u}}\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})}$ sufficiently small. So as in Danchin (2014), we use the estimates in the appendix therein and product laws in Besov spaces to get

$$\|f(\bar{\mathbf{u}})\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})} \le C\|\Delta \bar{\mathbf{u}}\|^2_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})} + CT\|(\rho_0 - 1, m_0 - 1)\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \tag{3.7}$$

Similarly, if both $\|\Delta \bar{\mathbf{u}}\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})}$ and $\|\Delta \bar{\mathbf{w}}\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})}$ are small, and if T is also small, it holds that

$$\|f(\bar{\mathbf{u}}) - f(\bar{\mathbf{w}})\|_{L^1_T(\dot{B}^{\frac{2}{p}-1}_{p,1})} \le \delta \|\bar{\mathbf{u}} - \bar{\mathbf{w}}\|_{E_p} \tag{3.8}$$

where δ is a small number.

Based on (3.7), (3.8) and Theorem 3.1, the standard contraction mapping theorem guarantees a unique solution $\bar{\mathbf{u}}$ of (3.6) (hence (3.5)) in $E_p(T)$ provided T is sufficiently small.

Step 4. Back to the Euler coordinates. We can go back to the Euler coordinates through the inverse of X, where X is defined by (3.3). This gives the existence part of Proposition 1.1. The uniqueness part can be proved by repeating Step 1-Step 3.

4 The Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2 in the following three subsections. To find the hidden dissipation of the system (1.2) and to avoid tedious calculations we may assume that $\gamma = 2$ (the case that $\gamma = 1$ is much more easier), since the other cases can be essentially reduced to this case. We introduce an unknown good function φ as

$$\varphi \stackrel{\text{def}}{=} P + \frac{1}{2}m^2 - \frac{3}{2}. \tag{4.1}$$

Direct calculations show that $(\varphi, \mathbf{u})$ satisfies

$$\begin{cases} \partial_t\varphi + 3\text{div}\,\mathbf{u} + \mathbf{u}\cdot\nabla\varphi + 2\varphi\,\text{div}\,\mathbf{u} = 0, \\ \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\text{div}\,\mathbf{u} + \nabla\varphi = \mathbf{F}(a, \mathbf{u}, \varphi), \\ \varphi|_{t=0} = \varphi_0 \stackrel{\text{def}}{=} a_0^2 + 2a_0 + \frac{1}{2}b_0^2 + b_0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{4.2}$$

where

$$\mathbf{F}(a, \mathbf{u}, \varphi) \stackrel{\text{def}}{=} I(a)\nabla\varphi - I(a)(\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\text{div}\,\mathbf{u}) \quad \text{with} \quad I(a) \stackrel{\text{def}}{=} \frac{a}{1+a}. \tag{4.3}$$

Throughout we make the assumption that

$$\sup_{t\in\mathbb{R}_+,\, x\in\mathbb{R}^2} |a(t,x)| \le \frac{1}{2} \tag{4.4}$$

which will enable us to use freely the composition estimate stated in Lemma 2.8. Note that as $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Condition (4.4) will be ensured by the fact that the constructed solution has small norm in $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$. Without loss of generality, we assume that $\mu = 1$ and $\lambda = 0$ in the following argument.

4.1 Low-Frequency Estimates

To study the coupling among a, φ and $\mathbb{Q}\mathbf{u}$, it is convenient to set

$$\mathbb{Q}\mathbf{u} = -\Lambda^{-1}\nabla d.$$

Since d and $\mathbb{Q}\mathbf{u} = \nabla\Delta^{-1}\text{div}\,\mathbf{u}$ can be converted into each other by a zeroth-order homogeneous Fourier multiplier, it suffices to bound d in order to control $\mathbb{Q}\mathbf{u}$. Now one can infer from (4.2) that

$$\begin{cases} \partial_t\varphi + 3\Lambda d = f_1, \\ \partial_t d - 2\Delta d - \Lambda\varphi = f_2 \end{cases} \tag{4.5}$$

where

$$f_1 \stackrel{\text{def}}{=} -\mathbf{u}\cdot\nabla\varphi - 2\varphi\,\mathrm{div}\,\mathbf{u}, \quad f_2 \stackrel{\text{def}}{=} \Lambda^{-1}\mathrm{div}\,(-(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathbf{F}(a,\mathbf{u},\varphi)).$$

In this subsection, we prove the following crucial lemma.

Lemma 4.1 *For any $t \geq 0$, there holds that*

$$\begin{aligned}&\|(\varphi^\ell, d^\ell)\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} + \|(\varphi^\ell, d^\ell)\|_{L^1_t(\dot{B}^2_{2,1})}\\ &\quad\lesssim \|(\varphi^\ell_0, d^\ell_0)\|_{\dot{B}^0_{2,1}} + \|((f_1)^\ell, (f_2)^\ell)\|_{L^1_t(\dot{B}^0_{2,1})}.\end{aligned} \tag{4.6}$$

Proof Let k_0 be some integer. Setting $f_k = \dot{\Delta}_k f$, applying the operator $\dot{\Delta}_k\dot{S}_{k_0}$ to the equations in (4.5), then multiplying $(4.5)_1$ by $\varphi^\ell_k/3$, $(4.5)_2$ by d^ℓ_k, respectively, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\varphi^\ell_k\|^2_{L^2}/3 + \|d^\ell_k\|^2_{L^2}\Big) + 2\|\Lambda d^\ell_k\|^2_{L^2} = \big\langle (f_1)^\ell_k, \varphi^\ell_k/3\big\rangle + \big\langle (f_2)^\ell_k, d^\ell_k\big\rangle \tag{4.7}$$

where we have used the following cancellation

$$\big\langle 3\Lambda d^\ell_k, \varphi^\ell_k/3\big\rangle - \big\langle \Lambda\varphi^\ell_k, d^\ell_k\big\rangle = 0. \tag{4.8}$$

To capture the dissipation of φ, we need to consider the time derivative of the mixed terms involved in $\big\langle d^\ell_k, \Lambda\varphi^\ell_k\big\rangle$

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\big\langle d^\ell_k, \Lambda\varphi^\ell_k\big\rangle + \|\Lambda\varphi^\ell_k\|^2_{L^2} - 3\|\Lambda d^\ell_k\|^2_{L^2}\\ &\quad= -2\big\langle \Delta d^\ell_k, \Lambda\varphi^\ell_k\big\rangle - \big\langle (f_2)^\ell_k, \Lambda\varphi^\ell_k\big\rangle - \big\langle \Lambda(f_1)^\ell_k, d^\ell_k\big\rangle.\end{aligned} \tag{4.9}$$

To eliminate the highest order terms on the right-hand sides of (4.9), we next estimate $\|\Lambda\varphi^\ell_k\|^2_{L^2}$. From (4.5), we have

$$\partial_t\Lambda\varphi^\ell_k + 3\Lambda^2 d^\ell_k = \Lambda(f_1)^\ell_k. \tag{4.10}$$

Testing (4.10) by $2\Lambda\varphi^\ell_k/3$ yields

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\varphi^\ell_k\|^2_{L^2} = \big\langle 2\Delta d^\ell_k, \Lambda\varphi^\ell_k\big\rangle + \big\langle (f_1)^\ell_k, 2\Lambda^2\varphi^\ell_k/3\big\rangle. \tag{4.11}$$

Denote

$$\mathscr{L}^2_k \stackrel{\text{def}}{=} 3\|\varphi^\ell_k\|^2_{L^2} + 9\|d^\ell_k\|^2_{L^2} - \big\langle d^\ell_k, \Lambda\varphi^\ell_k\big\rangle + \frac{2}{3}\|\Lambda\varphi^\ell_k\|^2_{L^2}.$$

Summing up (4.7)×9, (4.9), and (4.11), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_k^2+\frac{33}{2}\|\Lambda d_k^\ell\|_{L^2}^2+\frac{1}{2}\|\Lambda\varphi_k^\ell\|_{L^2}^2\\&\quad=3\big\langle(f_1)_k^\ell,a_k^\ell\big\rangle+9\big\langle(f_2)_k^\ell,d_k^\ell\big\rangle-\big\langle(f_2)_k^\ell,\Lambda\varphi_k^\ell\big\rangle-\big\langle\Lambda(f_1)_k^\ell,d_k^\ell\big\rangle+\frac{2}{3}\big\langle(f_1)_k^\ell,\Lambda^2\varphi_k^\ell\big\rangle.\end{aligned}\tag{4.12}$$

It's straightforward to deduce from the low-frequency cutoff and Young's inequality that

$$\mathscr{L}_k^2\approx\|(\varphi_k^\ell,\Lambda\varphi_k^\ell,d_k^\ell)\|_{L^2}^2\approx\|(\varphi_k^\ell,d_k^\ell)\|_{L^2}^2,$$

which leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_k^2+2^{2k}\mathscr{L}_k^2\lesssim\|((f_1)_k^\ell,(f_2)_k^\ell)\|_{L^2}\mathscr{L}_k.\tag{4.13}$$

Dividing by $\mathscr{L}_k$ formally on both hand sides of (4.13), and then integrating from 0 to t, we finally get desired estimate (4.6) by summing up over $k\le k_0$. This proves the lemma. □

From Lemma 4.1 and the definitions of φ, d, f_1 and f_2, the low-frequency part of $(\varphi,\mathbb{Q}\mathbf{u})$ can be bounded by

$$\begin{aligned}&\|(\varphi^\ell,\mathbb{Q}\mathbf{u}^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^0)}+\|(\varphi^\ell,\mathbb{Q}\mathbf{u}^\ell)\|_{L_t^1(\dot{B}_{2,1}^2)}\\&\quad\lesssim\|(\varphi_0^\ell,\mathbb{Q}\mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^0}+\|(\mathbf{u}\cdot\nabla\varphi)^\ell\|_{L_t^1(\dot{B}_{2,1}^0)}+\|(\varphi\mathrm{div}\,\mathbf{u})^\ell\|_{L_t^1(\dot{B}_{2,1}^0)}\\&\qquad+\|(\mathbf{u}\cdot\nabla\mathbf{u})^\ell\|_{L_t^1(\dot{B}_{2,1}^0)}+\|(\mathbf{F}(a,\mathbf{u},\varphi)^\ell\|_{L_t^1(\dot{B}_{2,1}^0)}.\end{aligned}\tag{4.14}$$

In the following, we estimate successively each of terms on the right-hand side of (4.14). To simplify the writing, we introduce the following notation:

$$\begin{aligned}&\mathscr{E}_\infty(t)\stackrel{\mathrm{def}}{=}\|(\varphi,\mathbb{Q}\mathbf{u})^\ell\|_{\dot{B}_{2,1}^0}+\|\varphi^h\|_{\dot{B}_{p,1}^{\frac{2}{p}}}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}},\\&\mathscr{E}_1(t)\stackrel{\mathrm{def}}{=}\|(\varphi,\mathbb{Q}\mathbf{u})^\ell\|_{\dot{B}_{2,1}^2}+\|\varphi^h\|_{\dot{B}_{p,1}^{\frac{2}{p}}}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}.\end{aligned}$$

First of all, in view of Lemma 2.6, there holds

$$
\begin{aligned}
&\|(\mathbf{u}\cdot\nabla\varphi)^\ell\|_{\dot B^0_{2,1}}+\|(\varphi\mathrm{div}\,\mathbf{u})^\ell\|_{\dot B^0_{2,1}}\\
&\quad\lesssim\|\mathbf{u}\|_{\dot B^{\frac2p}_{p,1}}(\|\varphi^\ell\|_{\dot B^1_{2,1}}+\|\varphi^h\|_{\dot B^{\frac2p}_{p,1}})+\|\varphi\|_{\dot B^{\frac2p}_{p,1}}(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^1_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac2p}_{p,1}})\\
&\quad\lesssim\|\mathbb{P}\mathbf{u}\|^2_{\dot B^{\frac2p}_{p,1}}+\|\mathbb{Q}\mathbf{u}^\ell\|^2_{\dot B^1_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|^2_{\dot B^{\frac2p}_{p,1}}+\|\varphi^\ell\|^2_{\dot B^1_{2,1}}+\|\varphi^h\|^2_{\dot B^{\frac2p}_{p,1}}\\
&\quad\lesssim\|\mathbb{P}\mathbf{u}\|_{\dot B^{\frac2p-1}_{p,1}}\|\mathbb{P}\mathbf{u}\|_{\dot B^{\frac2p+1}_{p,1}}+\|\varphi^h\|^2_{\dot B^{\frac2p}_{p,1}}\\
&\qquad+\Big(\|(\varphi^\ell,\mathbb{Q}\mathbf{u}^\ell)\|_{\dot B^0_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac2p-1}_{p,1}}\Big)\Big(\|(\varphi^\ell,\mathbb{Q}\mathbf{u}^\ell)\|_{\dot B^2_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac2p+1}_{p,1}}\Big)\\
&\quad\lesssim\mathscr{E}_\infty(t)\mathscr{E}_1(t).
\end{aligned}\tag{4.15}
$$

Next, to bound $\|(\mathbf{u}\cdot\nabla\mathbf{u})^\ell\|_{\dot B^0_{2,1}}$, we obtain from the decomposition $\mathbf{u}=\mathbb{P}\mathbf{u}+\mathbb{Q}\mathbf{u}$ and Lemma 2.6 that

$$
\begin{aligned}
\|(\mathbf{u}\cdot\nabla\mathbf{u})^\ell\|_{\dot B^0_{2,1}}&\lesssim\|(\mathbb{P}\mathbf{u}\cdot\nabla\mathbf{u})^\ell\|_{\dot B^0_{2,1}}+\|(\mathbb{Q}\mathbf{u}\cdot\nabla\mathbf{u})^\ell\|_{\dot B^0_{2,1}}\\
&\lesssim\|\mathbb{P}\mathbf{u}\|_{\dot B^{\frac2p-1}_{p,1}}\|\nabla\mathbf{u}\|_{\dot B^{\frac2p}_{p,1}}+(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^0_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac2p-1}_{p,1}})\|\nabla\mathbf{u}\|_{\dot B^{\frac2p}_{p,1}}.
\end{aligned}\tag{4.16}
$$

Due to

$$
\|\nabla\mathbf{u}\|_{\dot B^{\frac2p}_{p,1}}\lesssim\|\mathbb{P}\mathbf{u}\|_{\dot B^{\frac2p+1}_{p,1}}+\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^2_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac2p+1}_{p,1}}\lesssim\mathscr{E}_1(t),
$$

we infer from (4.16) that

$$
\|(\mathbf{u}\cdot\nabla\mathbf{u})^\ell\|_{\dot B^0_{2,1}}\lesssim\mathscr{E}_\infty(t)\mathscr{E}_1(t).\tag{4.17}
$$

We now turn to bound the terms involving composition functions in $\mathbf{F}(a,\mathbf{u},\varphi)$. Keeping in mind that

$$
I(a)=a-aI(a),
$$

we first use Lemmas 2.6 and 2.8 to get

$$\begin{aligned}\|(I(a))^\ell\|_{\dot B^0_{2,1}} &\lesssim \|a^\ell\|_{\dot B^0_{2,1}} + \|(aI(a))^\ell\|_{\dot B^0_{2,1}}\\ &\lesssim \|a^\ell\|_{\dot B^0_{2,1}} + \|I(a)\|_{\dot B^{\frac{2}{p}}_{p,1}}(\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}})\\ &\lesssim \|a^\ell\|_{\dot B^0_{2,1}} + \|a\|_{\dot B^{\frac{2}{p}}_{p,1}}(\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}})\\ &\lesssim \|a^\ell\|_{\dot B^0_{2,1}} + (\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}})^2\\ &\lesssim (1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t). \end{aligned} \tag{4.18}$$

Similarly, we can infer from Lemmas 2.5 and 2.8 that

$$\begin{aligned}\|I(a)\|_{\dot B^{\frac{2}{p}-1}_{p,1}} &\lesssim \|a\|_{\dot B^{\frac{2}{p}-1}_{p,1}} + \|aI(a)\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\\ &\lesssim (\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}}) + \|a\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|I(a)\|_{\dot B^{\frac{2}{p}}_{p,1}}\\ &\lesssim (\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}}) + \|a\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|a\|_{\dot B^{\frac{2}{p}}_{p,1}}\\ &\lesssim (1+\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}})(\|a^\ell\|_{\dot B^0_{2,1}} + \|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}})\\ &\lesssim (1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t). \end{aligned} \tag{4.19}$$

Now, for the first term $\|(I(a)\nabla\varphi)^\ell\|_{\dot B^0_{2,1}}$ in $\mathbf{F}(a,\mathbf{u},\varphi)$, in view of the fact that $\varphi = \varphi^\ell + \varphi^h$, we can write

$$\|(I(a)\nabla\varphi)^\ell\|_{\dot B^0_{2,1}} \lesssim \|(I(a)\nabla\varphi^\ell)^\ell\|_{\dot B^0_{2,1}} + \|(I(a)\nabla\varphi^h)^\ell\|_{\dot B^0_{2,1}}. \tag{4.20}$$

Thanks to Lemma 2.6 again, we have

$$\begin{aligned}\|(I(a)\nabla\varphi^\ell)^\ell\|_{\dot B^0_{2,1}} &\lesssim \|\nabla\varphi^\ell\|_{\dot B^{\frac{2}{p}}_{p,1}}(\|(I(a))^\ell\|_{\dot B^0_{2,1}} + \|(I(a))^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}})\\ &\lesssim \|\varphi^\ell\|_{\dot B^2_{2,1}}(\|(I(a))^\ell\|_{\dot B^0_{2,1}} + \|(I(a))^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}}) \end{aligned}$$

which combines (4.18) and (4.19) leads to

$$\|(I(a)\nabla\varphi^\ell)^\ell\|_{\dot B^0_{2,1}} \lesssim \|\varphi^\ell\|_{\dot B^2_{2,1}}(1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t). \tag{4.21}$$

For the term $\|(I(a)\nabla\varphi^h)^\ell\|_{\dot B^0_{2,1}}$ in (4.20), we use Bony's decomposition to write

$$\begin{aligned}\dot S_{j_0+1}(I(a)\nabla\varphi^h) =& \dot T_{I(a)}\dot S_{j_0+1}\nabla\varphi^h + [\dot S_{j_0+1}, \dot T_{I(a)}]\nabla\varphi^h\\ &+ \dot S_{j_0+1}\big(\dot T_{\nabla\varphi^h} I(a) + \dot R(I(a),\nabla\varphi^h)\big). \end{aligned} \tag{4.22}$$

Applying Lemma 2.3, there holds

$$\|\dot{T}_{I(a)}\dot{S}_{j_0+1}\nabla\varphi^h\|_{\dot{B}^0_{2,1}} \lesssim \|I(a)\|_{\dot{B}^{-1}_{\infty,\infty}}\|\dot{S}_{j_0+1}\nabla\varphi^h\|_{\dot{B}^1_{2,1}} \lesssim \|I(a)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\|\varphi\|^{\ell}_{\dot{B}^2_{2,1}}, \tag{4.23}$$

from which and (4.19), we can further get

$$\|\dot{T}_{I(a)}\dot{S}_{j_0+1}\nabla\varphi^h\|_{\dot{B}^0_{2,1}} \lesssim (1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t)\|\varphi\|^{\ell}_{\dot{B}^2_{2,1}}. \tag{4.24}$$

The last two terms in (4.22) can be estimated the same as (2.6) and (2.9) so that

$$\begin{aligned}
&\|\dot{S}_{j_0+1}\big(\dot{T}_{\nabla\varphi^h}I(a)+\dot{R}(I(a),\nabla\varphi^h)\big)\|_{\dot{B}^0_{2,1}} + \|[\dot{S}_{j_0+1},\dot{T}_{I(a)}]\nabla\varphi^h\|_{\dot{B}^0_{2,1}}\\
&\quad\lesssim \|\nabla\varphi^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\|I(a)\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\\
&\quad\lesssim \|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|a\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\\
&\quad\lesssim (\|a^\ell\|_{\dot{B}^0_{2,1}}+\|a^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}})\|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\\
&\quad\lesssim \mathscr{E}_\infty(t)\|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}
\end{aligned} \tag{4.25}$$

this together with (4.24) give rises to

$$\|(I(a)\nabla\varphi^h)^\ell\|_{\dot{B}^0_{2,1}} \lesssim (1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t)(\|\varphi\|^{\ell}_{\dot{B}^2_{2,1}}+\|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}). \tag{4.26}$$

Plugging (4.21) and (4.26) into (4.20) yields

$$\|(I(a)\nabla\varphi)\|_{\dot{B}^0_{2,1}} \lesssim (1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t)(\|\varphi\|^{\ell}_{\dot{B}^2_{2,1}}+\|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}). \tag{4.27}$$

For the last term in $\mathbf{F}(a,\mathbf{u},\varphi)$, as we set $\mathbb{P}\mathbf{u}$ in the L^p type spaces, we cannot use Lemma 2.6 directly to bound this term. For an integer $j_0 \geq 0$, we use Bony's decomposition to rewrite this term into

$$\begin{aligned}
\dot{S}_{j_0+1}\mathbb{Q}(I(a)\Delta\mathbf{u}) =&\dot{S}_{j_0+1}\mathbb{Q}\big(\dot{T}_{\Delta\mathbf{u}}I(a)+\dot{R}(\Delta\mathbf{u},I(a))\big)\\
&+\dot{T}_{I(a)}\Delta\dot{S}_{j_0+1}\mathbb{Q}\mathbf{u}+[\dot{S}_{j_0+1}\mathbb{Q},\dot{T}_{I(a)}]\Delta\mathbf{u}.
\end{aligned} \tag{4.28}$$

The first term can be bounded by Lemmas 2.3 and 2.8,

$$
\begin{aligned}
&\|(\dot{S}_{j_0+1}\mathbb{Q}\big(\dot{T}_{\Delta\mathbf{u}}I(a)+\dot{R}(\Delta\mathbf{u},I(a))\big))^{\ell}\|_{\dot{B}^0_{2,1}}\\
&\quad\lesssim \|I(a)\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\Delta\mathbf{u}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\lesssim \|a\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\mathbf{u}\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}\\
&\quad\lesssim (\|a^{\ell}\|_{\dot{B}^0_{2,1}}+\|a^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}})(\|\mathbb{Q}\mathbf{u}^{\ell}\|_{\dot{B}^2_{2,1}}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}).
\end{aligned}
\tag{4.29}
$$

Similarly, we have

$$
\begin{aligned}
\|(\dot{T}_{I(a)}\Delta\dot{S}_{j_0+1}\mathbb{Q}\mathbf{u})^{\ell}\|_{\dot{B}^0_{2,1}}\lesssim&\|I(a)\|_{L^\infty}\|\Delta\dot{S}_{j_0+1}\mathbb{Q}\mathbf{u}\|_{\dot{B}^0_{2,1}}\\
\lesssim&\|a\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\mathbb{Q}\mathbf{u}^{\ell}\|_{\dot{B}^2_{2,1}}\\
\lesssim&(\|a^{\ell}\|_{\dot{B}^0_{2,1}}+\|a^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}})\|\mathbb{Q}\mathbf{u}^{\ell}\|_{\dot{B}^2_{2,1}}.
\end{aligned}
\tag{4.30}
$$

The commutator term is estimated by using Lemma 6.1 in Danchin and He (2016) that

$$
\begin{aligned}
\|[\dot{S}_{j_0+1}\mathbb{Q},T_{I(a)}]\Delta\mathbf{u}\|_{\dot{B}^0_{2,1}}\lesssim&\|\nabla I(a)\|_{\dot{B}^{\frac{2}{p^*}-1}_{p^*,1}}\|\nabla^2\mathbf{u}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}},\qquad\left(\frac{1}{p^*}+\frac{1}{p}=\frac{1}{2}\right),\\
\lesssim&\|\nabla I(a)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\|\mathbf{u}\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}\\
\lesssim&\|a\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\mathbf{u}\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}\\
\lesssim&(\|a^{\ell}\|_{\dot{B}^0_{2,1}}+\|a^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}})(\|\mathbb{Q}\mathbf{u}^{\ell}\|_{\dot{B}^2_{2,1}}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}),
\end{aligned}
\tag{4.31}
$$

where we have used the embedding $\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)\hookrightarrow\dot{B}^{\frac{2}{p^*}-1}_{p^*,1}(\mathbb{R}^2),\ 2\le p<4.$

The term $I(a)\nabla\mathrm{div}\,\mathbf{u}$ can be estimated in a similar manner. As a result, we have

$$
\|(I(a)(\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u}))^{\ell}\|_{\dot{B}^0_{2,1}}\lesssim\mathscr{E}_\infty(t)\mathscr{E}_1(t).
\tag{4.32}
$$

Plugging (4.15), (4.17), (4.27), and (4.32) into (4.14) gives

$$
\begin{aligned}
&\|(\varphi^{\ell},\mathbb{Q}\mathbf{u}^{\ell})\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})}+\|(\varphi^{\ell},\mathbb{Q}\mathbf{u}^{\ell})\|_{L^1_t(\dot{B}^2_{2,1})}\\
&\quad\lesssim\|(\varphi_0^{\ell},\mathbb{Q}\mathbf{u}_0^{\ell})\|_{\dot{B}^0_{2,1}}+\int_0^t(1+\mathscr{E}_\infty(\tau))\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau.
\end{aligned}
\tag{4.33}
$$

Finally, we shall derive the bound of $\|a^{\ell}\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})}$. Due to the appearance of the term $\mathrm{div}\,\mathbf{u}$ in the first equation of (1.3), we cannot obtain the bound $\|a^{\ell}\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})}$ directly.

To break the barrier, we define

$$\delta \stackrel{\text{def}}{=} \varphi - 3a \tag{4.34}$$

which satisfies the following transport equation

$$\partial_t \delta + \mathbf{u} \cdot \nabla \delta + \delta \operatorname{div} \mathbf{u} + \varphi \operatorname{div} \mathbf{u} = 0. \tag{4.35}$$

Now applying $\dot{\Delta}_j$ to the above equation and using a commutator's argument give rise to

$$\partial_t \dot{\Delta}_j \delta + \mathbf{u} \cdot \nabla \dot{\Delta}_j \delta + [\dot{\Delta}_j, \mathbf{u} \cdot \nabla]\delta + \dot{\Delta}_j(\delta \operatorname{div} \mathbf{u}) + \dot{\Delta}_j(\varphi \operatorname{div} \mathbf{u}) = 0.$$

Taking L^2 inner product of the resulting equation with $\dot{\Delta}_j \delta$, applying the Hölder inequality and integrating the resultant inequality over $[0, t]$, then summing up $j \leq j_0$, we arrive at

$$\begin{aligned} \|\delta^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} \lesssim & \|\delta^\ell_0\|_{\dot{B}^0_{2,1}} + \|(\delta \operatorname{div} \mathbf{u})^\ell\|_{L^1_t(\dot{B}^0_{2,1})} + \|(\varphi \operatorname{div} \mathbf{u})^\ell\|_{L^1_t(\dot{B}^0_{2,1})} \\ & + \int_0^t \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\delta^\ell\|_{\dot{B}^0_{2,1}} \, d\tau + \int_0^t \sum_{j \leq j_0} \|[\dot{\Delta}_j, \mathbf{u} \cdot \nabla]\delta\|_{L^2} \, d\tau. \end{aligned} \tag{4.36}$$

By Lemma 2.6, there holds

$$\begin{aligned} \|(\delta \operatorname{div} \mathbf{u})^\ell\|_{\dot{B}^0_{2,1}} \lesssim & (\|\delta^\ell\|_{\dot{B}^0_{2,1}} + \|\delta^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}) \|\operatorname{div} \mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \\ \lesssim & (\|(a^\ell, \varphi^\ell)\|_{\dot{B}^0_{2,1}} + \|(a^h, \varphi^h)\|_{\dot{B}^{\frac{2}{p}}_{p,1}})(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot{B}^2_{2,1}} + \|\mathbb{Q}\mathbf{u}^h\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}). \end{aligned} \tag{4.37}$$

Similarly,

$$\|(\varphi \operatorname{div} \mathbf{u})^\ell\|_{\dot{B}^0_{2,1}} \lesssim (\|\varphi^\ell\|_{\dot{B}^0_{2,1}} + \|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}})(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot{B}^2_{2,1}} + \|\mathbb{Q}\mathbf{u}^h\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}). \tag{4.38}$$

With the aid of the embedding relation $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and Lemma 2.7, we can bound the forth term on the right-hand side of (4.36) as

$$\begin{aligned} \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\delta^\ell\|_{\dot{B}^0_{2,1}} \lesssim & \|\delta^\ell\|_{\dot{B}^0_{2,1}} \|\mathbb{Q}\mathbf{u}\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}} \\ \lesssim & (\|(a^\ell, \varphi^\ell)\|_{\dot{B}^0_{2,1}} + \|(a^h, \varphi^h)\|_{\dot{B}^{\frac{2}{p}}_{p,1}})(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot{B}^2_{2,1}} + \|\mathbb{Q}\mathbf{u}^h\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}}). \end{aligned} \tag{4.39}$$

The last term in (4.36) can be bounded by a similarly derivation of (4.9) in He et al. (2019) that

$$\sum_{j\le j_0}\|[\dot\Delta_j,\mathbf{u}\cdot\nabla]\delta\|_{L^2}\lesssim(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^2_{2,1}}+\|(\mathbb{P}\mathbf{u},\mathbf{u}^h)\|_{\dot B^{\frac{2}{p}+1}_{p,1}})(\|(a^\ell,\varphi^\ell)\|_{\dot B^0_{2,1}} + \|(a^h,\varphi^h)\|_{\dot B^{\frac{2}{p}}_{p,1}}). \tag{4.40}$$

Taking (4.37)–(4.40) into (4.36), we obtain

$$\|\delta^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}\lesssim\|(a^\ell_0,\varphi^\ell_0)\|_{\dot B^0_{2,1}}+\int_0^t\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau \tag{4.41}$$

which combines the definition $a=\frac{1}{3}(\varphi-\delta)$ leads to

$$\begin{aligned}\|a^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}&\lesssim\|\delta^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}+\|\varphi^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}\\&\lesssim\|(a^\ell_0,\varphi^\ell_0)\|_{\dot B^0_{2,1}}+\|\varphi^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}+\int_0^t\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{4.42}$$

In the same manner, we can infer from forth equation of (1.3) that

$$\|b^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}\lesssim\|(b^\ell_0,\varphi^\ell_0)\|_{\dot B^0_{2,1}}+\|\varphi^\ell\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}+\int_0^t\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau. \tag{4.43}$$

Consequently, combining with (4.33), (4.42) and (4.43), we finally arrive at

$$\begin{aligned}&\|(a^\ell,b^\ell,\mathbb{Q}\mathbf{u}^\ell)\|_{\widetilde L^\infty_t(\dot B^0_{2,1})}+\|(\varphi^\ell,\mathbb{Q}\mathbf{u}^\ell)\|_{L^1_t(\dot B^2_{2,1})}\\&\quad\lesssim\|(a^\ell_0,b^\ell_0,\varphi^\ell_0,\mathbb{Q}\mathbf{u}^\ell_0)\|_{\dot B^0_{2,1}}+\int_0^t(1+\mathscr{E}_\infty(\tau))\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{4.44}$$

4.2 High-Frequency Estimates

In this subsection, we shall introduce the so-called effective velocity to capture the damping effect of φ in the high-frequency part.

4.2.1 Estimates for Auxiliary Unknowns

First, we infer from (4.2) that $(\varphi,\mathbb{Q}\mathbf{u})$ satisfies

$$\begin{cases}\partial_t\varphi+3\mathrm{div}\,\mathbf{u}=-\varphi\,\mathrm{div}\,\mathbf{u}-\mathrm{div}\,(\varphi\mathbf{u}),\\ \partial_t\mathbb{Q}\mathbf{u}-2\Delta\mathbb{Q}\mathbf{u}+\nabla\varphi=-\mathbb{Q}(\mathbf{u}\cdot\nabla\mathbf{u})+\mathbb{Q}\mathbf{F}(a,\mathbf{u},\varphi).\end{cases} \tag{4.45}$$

Now, we define the effective velocity $\mathbf{G}$ as follows

$$\mathbf{G} \stackrel{\text{def}}{=} \mathbb{Q}\mathbf{u} - \frac{1}{2}\Delta^{-1}\nabla\varphi. \tag{4.46}$$

Then $\mathbf{G}$ satisfies

$$\begin{aligned} \partial_t \mathbf{G} - 2\Delta\mathbf{G} = \frac{3}{2}\mathbf{G} + \frac{3}{2}\Delta^{-1}\nabla\varphi + \frac{1}{2}\Delta^{-1}\nabla(\varphi \operatorname{div} \mathbf{u}) \\ + \frac{1}{2}\mathbb{Q}(\varphi\mathbf{u}) - \mathbb{Q}(\mathbf{u}, \nabla\mathbf{u}) + \mathbb{Q}\mathbf{F}(a, \mathbf{u}, \varphi). \end{aligned} \tag{4.47}$$

Applying the heat estimate (2.9) for the high frequencies of $\mathbf{G}$ only, we get

$$\begin{aligned} &\|\mathbf{G}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \\ &\quad \lesssim \|\mathbf{G}_0^h\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\varphi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-2})} + \|(\varphi \operatorname{div} \mathbf{u})^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-2})} \\ &\qquad + \|\mathbb{Q}(\varphi\mathbf{u})^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\mathbb{Q}(\mathbf{u}, \nabla\mathbf{u})^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\mathbb{Q}\mathbf{F}(a, \mathbf{u}, \varphi)^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}. \end{aligned} \tag{4.48}$$

The important point is that, owing to the high-frequency cutoff at $|\xi| \sim 2^{j_0}$,

$$\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim 2^{-2j_0}\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \quad \text{and} \quad \|\varphi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-2})} \lesssim 2^{-2j_0}\|\varphi^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}.$$

Hence, if j_0 is large enough, then the term $\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}$ may be absorbed by the right-hand side.

In view of (4.46), we have φ satisfies

$$\partial_t\varphi + \frac{3}{2}\varphi + \mathbf{u} \cdot \nabla\varphi = -3\operatorname{div} \mathbf{G} - 2\varphi \operatorname{div} \mathbf{u}. \tag{4.49}$$

Applying $\dot{\Delta}_j$ to (4.49) and using a commutator argument give rise to

$$\partial_t \dot{\Delta}_j\varphi + \frac{3}{2}\dot{\Delta}_j\varphi + \mathbf{u} \cdot \nabla\dot{\Delta}_j\varphi = -[\dot{\Delta}_j, \mathbf{u} \cdot \nabla]\varphi - 3\dot{\Delta}_j\operatorname{div} \mathbf{G} - 2\dot{\Delta}_j(\varphi \operatorname{div} \mathbf{u}). \tag{4.50}$$

Taking L^2 inner product of (4.50) with $\frac{1}{p}|\dot{\Delta}_j\varphi|^{p-2}\dot{\Delta}_j\varphi$, applying the Hölder inequality and integrating the resultant inequality over $[0, t]$ lead to

$$\begin{aligned}
&\|\dot{\Delta}_j\varphi(t)\|_{L^p} + \frac{3}{2}\int_0^t \|\dot{\Delta}_j\varphi\|_{L^p}\,\mathrm{d}\tau \\
&\lesssim \|\dot{\Delta}_j\varphi_0\|_{L^p} + \frac{1}{p}\int_0^t \|\mathrm{div}\,\mathbf{u}\|_{L^\infty}\|\dot{\Delta}_j\varphi\|_{L^p}\,\mathrm{d}\tau \\
&\quad + \int_0^t \|[\dot{\Delta}_j, \mathbf{u}\cdot\nabla]\varphi\|_{L^p}\,\mathrm{d}\tau + 3\int_0^t \|\dot{\Delta}_j \mathrm{div}\,\mathbf{G}\|_{L^p}\,\mathrm{d}\tau + \int_0^t \|\dot{\Delta}_j(\varphi\,\mathrm{div}\,\mathbf{u})\|_{L^p}\,\mathrm{d}\tau
\end{aligned} \tag{4.51}$$

from which we can further get

$$\begin{aligned}
&\|\varphi^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})} + \frac{3}{2}\|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})} \\
&\quad\lesssim \|\varphi_0^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + 3\|\mathbf{G}^h\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\varphi\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\,\mathrm{d}\tau.
\end{aligned} \tag{4.52}$$

Multiplying (4.48) by a suitable large constant and adding to (4.52), we obtain

$$\begin{aligned}
&\|\mathbf{G}^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} + \|\varphi^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})} + \|\mathbf{G}^h\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})} \\
&\lesssim \|\mathbf{G}_0^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \|\varphi_0^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \int_0^t \|\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\varphi\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\,\mathrm{d}\tau \\
&\quad + \int_0^t (\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \|(\varphi\mathbf{u})^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}})\,\mathrm{d}\tau + \int_0^t \|(\mathbf{F}(a,\mathbf{u},\varphi))^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\,\mathrm{d}\tau.
\end{aligned} \tag{4.53}$$

4.2.2 Recovering Estimates for a, b

In this subsection, we shall recover the estimates for a, b, and $\mathbf{u}$. On the one hand, in view of $\mathbf{G} \overset{\mathrm{def}}{=} \mathbb{Q}\mathbf{u} - \frac{1}{2}\Delta^{-1}\nabla\varphi$ and the embedding relation in the high frequency, there holds

$$\begin{aligned}
\|\mathbb{Q}\mathbf{u}^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} &\lesssim \|\mathbf{G}^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} + \|\varphi^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}, \\
\|\mathbb{Q}\mathbf{u}^h\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} &\lesssim \|\mathbf{G}^h\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})}.
\end{aligned}$$

As a result, we can rewrite (4.53) into

$$
\begin{aligned}
&\|\varphi^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}+\|\mathbb{Q}\mathbf{u}^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})}+\|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})}+\|\mathbb{Q}\mathbf{u}^h\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})}\\
&\quad\lesssim\|\varphi^h_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}}+\|\mathbb{Q}\mathbf{u}^h_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\int_0^t\|\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\varphi\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\,\mathrm{d}\tau\\
&\qquad+\int_0^t(\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\|(\varphi\mathbf{u})^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}})\,\mathrm{d}\tau+\int_0^t\|(\mathbf{F}(a,\mathbf{u},\varphi))^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\,\mathrm{d}\tau.
\end{aligned}
\tag{4.54}
$$

Finally, we estimate the incompressible part of the velocity field. Applying the operator $\mathbb{P}$ to the second equation of (4.2), we find that $\mathbb{P}\mathbf{u}$ satisfies the heat equation

$$
\partial_t\mathbb{P}\mathbf{u}-\Delta\mathbb{P}\mathbf{u}=-\mathbb{P}(\mathbf{u}\cdot\nabla\mathbf{u})+\mathbb{P}\mathbf{F}(a,\mathbf{u},\varphi).
\tag{4.55}
$$

By Lemma 2.9, we can get

$$
\begin{aligned}
&\|\mathbb{P}\mathbf{u}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})}+\|\mathbb{P}\mathbf{u}\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})}\\
&\quad\lesssim\|\mathbb{P}\mathbf{u}_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\|(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^1_t(\dot{B}^{\frac{2}{p}-1}_{p,1})}+\|(\mathbf{F}(a,\mathbf{u},\varphi)\|_{L^1_t(\dot{B}^{\frac{2}{p}-1}_{p,1})}.
\end{aligned}
\tag{4.56}
$$

Combining (4.54) with (4.56) gives

$$
\begin{aligned}
&\|\varphi^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})}\\
&\qquad+\|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})}\\
&\quad\lesssim\|\varphi^h_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}}+\|(\mathbb{P}\mathbf{u}_0,\mathbb{Q}\mathbf{u}^h_0)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\int_0^t\|\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\varphi\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\,\mathrm{d}\tau\\
&\qquad+\int_0^t(\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\|(\varphi\mathbf{u})^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}})\,\mathrm{d}\tau+\int_0^t\|\mathbf{F}(a,\mathbf{u},\varphi)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\,\mathrm{d}\tau.
\end{aligned}
\tag{4.57}
$$

We now bound the terms on the right-hand side of (4.57). First, it's obvious that

$$
\|\nabla\mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\|\varphi\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\lesssim\big(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot{B}^2_{2,1}}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}})(\|\varphi^\ell\|_{\dot{B}^0_{2,1}}+\|\varphi^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\big).
\tag{4.58}
$$

Then, according to Lemma 2.5, there holds

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot B^{\frac{2}{p}-1}_{p,1}} \lesssim&\|\mathbf{u}\|^2_{\dot B^{\frac{2}{p}}_{p,1}}\\ \lesssim&\|\mathbb{P}\mathbf{u}\|^2_{\dot B^{\frac{2}{p}}_{p,1}}+\|\mathbb{Q}\mathbf{u}^\ell\|^2_{\dot B^1_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|^2_{\dot B^{\frac{2}{p}}_{p,1}}\\ \lesssim&\|\mathbb{P}\mathbf{u}\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|\mathbb{P}\mathbf{u}\|_{\dot B^{\frac{2}{p}+1}_{p,1}}+\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^0_{2,1}}\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^2_{2,1}}+\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|\mathbb{Q}\mathbf{u}^h\|_{\dot B^{\frac{2}{p}+1}_{p,1}}\\ \lesssim&\mathscr{E}_\infty(t)\mathscr{E}_1(t).\end{aligned} \tag{4.59}$$

By the embedding relation in high frequency and the Young inequality, we get

$$\|(\varphi\mathbf{u})^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\lesssim\|\varphi\mathbf{u}\|_{\dot B^{\frac{2}{p}}_{p,1}}\lesssim\|\varphi\|^2_{\dot B^{\frac{2}{p}}_{p,1}}+\|\mathbf{u}\|^2_{\dot B^{\frac{2}{p}}_{p,1}}\lesssim\mathscr{E}_\infty(t)\mathscr{E}_1(t). \tag{4.60}$$

At last, we deal with each term in $\mathbf{F}(a,\mathbf{u},\varphi)$. In view of Lemmas 2.5, 2.8, there holds

$$\begin{aligned}\|I(a)\nabla\varphi\|_{\dot B^{\frac{2}{p}-1}_{p,1}} \lesssim&\|I(a)\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|\nabla\varphi^\ell\|_{\dot B^{\frac{2}{p}}_{p,1}}+\|I(a)\|_{\dot B^{\frac{2}{p}}_{p,1}}\|\nabla\varphi^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\\ \lesssim&\|I(a)\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|\varphi^\ell\|_{\dot B^2_{2,1}}+\|a\|_{\dot B^{\frac{2}{p}}_{p,1}}\|\varphi^h\|_{\dot B^{\frac{2}{p}}_{p,1}}\\ \lesssim&\|I(a)\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\|\varphi^\ell\|_{\dot B^2_{2,1}}+(\|a^\ell\|_{\dot B^0_{2,1}}+\|a^h\|_{\dot B^{\frac{2}{p}}_{p,1}})\|\varphi^h\|_{\dot B^{\frac{2}{p}}_{p,1}}\end{aligned} \tag{4.61}$$

from which and (4.19), we can further get

$$\|I(a)\nabla\varphi\|_{\dot B^{\frac{2}{p}-1}_{p,1}}\lesssim(1+\mathscr{E}_\infty(t))\mathscr{E}_\infty(t)(\|\varphi\|^\ell_{\dot B^2_{2,1}}+\|\varphi^h\|_{\dot B^{\frac{2}{p}}_{p,1}}). \tag{4.62}$$

The last term in $\mathbf{F}(a,\mathbf{u},\varphi)$ can be bounded in the same manner. Hence, we have

$$\begin{aligned}\|(I(a)(\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u}))^h\|_{\dot B^{\frac{2}{p}-1}_{p,1}} \lesssim&\|I(a)\|_{\dot B^{\frac{2}{p}}_{p,1}}\|\mathbf{u}\|_{\dot B^{\frac{2}{p}+1}_{p,1}}\\ \lesssim&\|a\|_{\dot B^{\frac{2}{p}}_{p,1}}(\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot B^2_{2,1}}+\|(\mathbb{P}\mathbf{u}^h,\mathbb{Q}\mathbf{u}^h)\|_{\dot B^{\frac{2}{p}+1}_{p,1}})\\ \lesssim&\mathscr{E}_\infty(t)\mathscr{E}_1(t).\end{aligned} \tag{4.63}$$

Collecting the estimates above, we get from (4.57) that

$$\begin{aligned}&\|\varphi^h\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}}_{p,1})}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}-1}_{p,1})}\\ &\quad+\|\varphi^h\|_{L^1_t(\dot B^{\frac{2}{p}}_{p,1})}+\|(\mathbb{P}\mathbf{u},\mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot B^{\frac{2}{p}+1}_{p,1})}\\ &\lesssim\|\varphi^h_0\|_{\dot B^{\frac{2}{p}}_{p,1}}+\|(\mathbb{P}\mathbf{u}_0,\mathbb{Q}\mathbf{u}^h_0)\|_{\dot B^{\frac{2}{p}-1}_{p,1}}+\int_0^t(1+\mathscr{E}_\infty(\tau))\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{4.64}$$

For the term $\|a^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}$, we get by a similar derivation of (4.52) that

$$\begin{aligned}
\|a^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})} &\lesssim \|a^h_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \|\mathbf{u}^h\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \int_0^t \|\nabla \mathbf{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \,\mathrm{d}\tau \\
&\lesssim \|a^h_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} \\
&\quad + \int_0^t (\|\mathbb{Q}\mathbf{u}^\ell\|_{\dot{B}^2_{2,1}} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{\dot{B}^{\frac{2}{p}+1}_{p,1}})(\|a^\ell\|_{\dot{B}^0_{2,1}} + \|a^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}) \,\mathrm{d}\tau \\
&\lesssim \|a^h_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \int_0^t \mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau. \qquad (4.65)
\end{aligned}$$

In the same manner, we can infer that

$$\|b^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})} \lesssim \|b^h_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} + \int_0^t \mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau. \qquad (4.66)$$

Multiplying by a suitable large constant on both sides of (4.64) and then pulsing (4.65) and (4.66), we can finally get

$$\begin{aligned}
&\|(a^h, b^h)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} \\
&\quad + \|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})} \\
&\lesssim \|(a^h_0, b^h_0, \varphi^h_0)\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \|(\mathbb{P}\mathbf{u}_0, \mathbb{Q}\mathbf{u}^h_0)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} + \int_0^t (1 + \mathscr{E}_\infty(\tau))\mathscr{E}_\infty(\tau)\mathscr{E}_1(\tau)\,\mathrm{d}\tau.
\end{aligned} \qquad (4.67)$$

4.3 Proof of Theorem 1.2

In this subsection, we shall give the proof of Theorem 1.2 by the local existence result and the continuation argument. Denote

$$\begin{aligned}
\mathscr{X}(t) &\stackrel{\text{def}}{=} \|(a^\ell, \mathbb{Q}\mathbf{u}^\ell, b^\ell)\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} + \|(a^h, b^h)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}-1}_{p,1})} \\
&\quad + \|(\varphi^\ell, \mathbb{Q}\mathbf{u}^\ell)\|_{L^1_t(\dot{B}^2_{2,1})} + \|\varphi^h\|_{L^1_t(\dot{B}^{\frac{2}{p}}_{p,1})} + \|(\mathbb{P}\mathbf{u}, \mathbb{Q}\mathbf{u}^h)\|_{L^1_t(\dot{B}^{\frac{2}{p}+1}_{p,1})}, \\
\mathscr{X}_0 &\stackrel{\text{def}}{=} \|(a^\ell_0, \mathbb{Q}\mathbf{u}^\ell_0, b^\ell_0)\|_{\dot{B}^0_{2,1}} + \|(a^h_0, b^h_0)\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + \|(\mathbb{P}\mathbf{u}_0, \mathbb{Q}\mathbf{u}^h_0)\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}. \qquad (4.68)
\end{aligned}$$

It follows from Lemmas 2.6 and 2.5 that

$$\begin{aligned}\|\varphi_0^\ell\|_{\dot{B}^0_{2,1}}+\|\varphi_0^h\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\lesssim&\|(a_0^\ell,b_0^\ell)\|_{\dot{B}^0_{2,1}}+\|(a_0^h,b_0^h)\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\\&+\|(a_0^\ell,b_0^\ell)\|_{\dot{B}^0_{2,1}}(\|(a_0^\ell,b_0^\ell)\|_{\dot{B}^0_{2,1}}+\|(a_0^h,b_0^h)\|_{\dot{B}^{\frac{2}{p}}_{p,1}})\\\lesssim&(1+\mathscr{X}_0)\mathscr{X}_0.\end{aligned}\tag{4.69}$$

Now, summing up (4.44) and (4.67), we get

$$\mathscr{X}(t)\le(1+\mathscr{X}_0)\mathscr{X}_0+C(\mathscr{X}(t))^2(1+C\mathscr{X}(t)).\tag{4.70}$$

Under the setting of initial data in Theorem 1.2, there exists a positive constant C_0 such that $\mathscr{X}_0\le C_0\epsilon$. Due to the local existence result which has been achieved by Proposition 1.1, there exists a positive time T such that

$$\mathscr{X}(t)\le 2C_0\,\epsilon,\quad\forall\, t\in[0,T].\tag{4.71}$$

Let T^* be the largest possible time of T for what (4.71) holds. Now, we only need to show $T^*=\infty$. By the estimate of (4.70), we can use a standard continuation argument to prove that $T^*=\infty$ provided that ϵ is small enough. We omit the details here. This finishes the proof of Theorem 1.2. □

5 The Proof of Theorem 1.3

In this section, we shall follow the method (independent of the spectral analysis) used in Guo and Wang (2012) and (Xin and Xu 2021) to get the decay rate of the solutions constructed in the previous section. From the proof of Theorem 1.2, we can get the following inequality (see the derivation of (4.33) and (4.64) for more details):

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|(\varphi,\mathbf{u})\|^\ell_{\dot{B}^0_{2,1}}+\|\varphi\|^h_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^h_{\dot{B}^0_{2,1}})+\|(\varphi,\mathbf{u})\|^\ell_{\dot{B}^2_{2,1}}+\|\varphi\|^h_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^h_{\dot{B}^2_{2,1}}\\&\quad\lesssim(1+\|(a,\mathbf{u},b)\|^\ell_{\dot{B}^0_{2,1}}+\|(a,b)\|^h_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^h_{\dot{B}^0_{2,1}})\\&\qquad\times(\|(a,\mathbf{u},b)\|^\ell_{\dot{B}^0_{2,1}}+\|(a,b)\|^h_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^h_{\dot{B}^0_{2,1}})(\|(\varphi,\mathbf{u})\|^\ell_{\dot{B}^2_{2,1}}+\|\varphi\|^h_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^h_{\dot{B}^2_{2,1}}).\end{aligned}\tag{5.1}$$

By Theorem 1.2,

$$\|(a,\mathbf{u},b)\|^\ell_{\dot{B}^0_{2,1}}+\|(a,b)\|^h_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^h_{\dot{B}^0_{2,1}}\le c_0.\tag{5.2}$$

Inserting (5.2) into (5.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|(\varphi,\mathbf{u})\|^{\ell}_{\dot{B}^0_{2,1}}+\|\varphi\|^{h}_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^{h}_{\dot{B}^0_{2,1}})+\bar{c}(\|(\varphi,\mathbf{u})\|^{\ell}_{\dot{B}^2_{2,1}}+\|\varphi\|^{h}_{\dot{B}^1_{2,1}}+\|\mathbf{u}\|^{h}_{\dot{B}^2_{2,1}})\le 0. \tag{5.3}$$

In order to derive the decay estimate of the solutions given in Theorem 1.2, we need to get a Lyapunov-type differential inequality from (5.3). According to (5.2) and the embedding relation in the high frequency, it's obvious for any $\beta>0$ that

$$\|\varphi\|^{h}_{\dot{B}^1_{2,1}}\ge C\big(\|\varphi\|^{h}_{\dot{B}^1_{2,1}}\big)^{1+\beta}, \tag{5.4}$$

and

$$\|\mathbf{u}\|^{h}_{\dot{B}^2_{2,1}}\ge C(\|\mathbf{u}\|^{h}_{\dot{B}^0_{2,1}})^{1+\beta}. \tag{5.5}$$

Thus, to get the Lyapunov-type inequality of the solutions, we only need to control the norm of $\|\varphi\|^{\ell}_{\dot{B}^2_{2,1}}$. This process can be obtained from the fact that the solutions constructed in Theorem 1.2 can propagate the regularity of the initial data in Besov space with low regularity, see the following Proposition 5.1. This will ensure that one can use interpolation to get the desired Lyapunov-type inequality.

Proposition 5.1 *Let $(a,\mathbf{u},b)$ be the solutions constructed in Theorem 1.2 with $p=2$. Assume further that $(a^{\ell}_0,\mathbf{u}^{\ell}_0,b^{\ell}_0)\in\dot{B}^{-\sigma}_{2,\infty}(\mathbb{R}^2)$ for some $0<\sigma\le 1$. Then there exists a constant $C_0>0$ depending on the norm of the initial data such that for all $t\ge 0$,*

$$\|(a,b,\mathbf{u},\varphi)(t,\cdot)\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}\le C_0. \tag{5.6}$$

Proof It follows from (4.7) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\varphi^{\ell}_k\|^2_{L^2}/3+\|d^{\ell}_k\|^2_{L^2}\Big)+2\|\Lambda d^{\ell}_k\|^2_{L^2}=\big\langle (f_1)^{\ell}_k,\varphi^{\ell}_k/3\big\rangle+\big\langle (f_2)^{\ell}_k,d^{\ell}_k\big\rangle \tag{5.7}$$

By performing a routine procedure, one obtains

$$\|(\varphi,\mathbf{u})\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}\lesssim\|(\varphi_0,\mathbf{u}_0)\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}+\int_0^t\|(f_1,f_2)\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau. \tag{5.8}$$

To control $\|a\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}$, we first get by taking $\dot{\Delta}_j$ on both hand sides of (4.35) that

$$\partial_t\dot{\Delta}_j\delta+\mathbf{u}\cdot\nabla\dot{\Delta}_j\delta+[\dot{\Delta}_j,\mathbf{u}\cdot\nabla]\delta=\dot{\Delta}_j f_3 \tag{5.9}$$

with $f_3\overset{\text{def}}{=}\delta\,\mathrm{div}\,\mathbf{u}+\varphi\,\mathrm{div}\,\mathbf{u}$.

Then, we get by a similar derivation of (4.36) that

$$\|\delta^\ell\|_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|\delta_0^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t \|\mathrm{div}\,\mathbf{u}\|_{L^\infty}\|\delta^\ell\|_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau + \int_0^t \|\nabla\mathbf{u}\|_{\dot B^1_{2,1}}\|\delta\|_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau + \int_0^t \|f_3\|_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau \tag{5.10}$$

in which we have used the Lemma 2.100 of Bahouri et al. (2011) to deal with the commutator.

With the aid of the embedding relation $\dot B^1_{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and Lemma 2.7, we have

$$\begin{aligned}\|\mathrm{div}\,\mathbf{u}\|_{L^\infty}\|\delta^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \|\nabla\mathbf{u}\|_{\dot B^1_{2,1}}\|\delta\|_{\dot B^{-\sigma}_{2,\infty}} &\lesssim \|\nabla\mathbf{u}\|_{\dot B^1_{2,1}}(\|\delta^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \|\delta^h\|_{\dot B^1_{2,1}})\\ &\lesssim \|\mathbf{u}\|_{\dot B^2_{2,1}}(\|\delta^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \|\delta^h\|_{\dot B^1_{2,1}})\\ &\lesssim \|\mathbf{u}\|_{\dot B^2_{2,1}}(\|(a^\ell,\varphi^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \|(a^h,\varphi^h)\|_{\dot B^1_{2,1}}).\end{aligned} \tag{5.11}$$

For the las term in (5.10), we use Lemma 2.4 directly to get

$$\begin{aligned}\|f_3\|_{\dot B^{-\sigma}_{2,\infty}} &\lesssim \|\nabla\mathbf{u}\|_{\dot B^1_{2,1}}\|(\delta,\varphi)\|_{\dot B^{-\sigma}_{2,\infty}}\\ &\lesssim \|\mathbf{u}\|_{\dot B^2_{2,1}}(\|(a^\ell,\varphi^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \|(a^h,\varphi^h)\|_{\dot B^1_{2,1}}).\end{aligned} \tag{5.12}$$

Inserting (5.11) and (5.12) into (5.10) yields

$$\|\delta^\ell\|_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|\delta_0^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t \|\mathbf{u}\|_{\dot B^2_{2,1}}(\|(a^\ell,\varphi^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \|(a^h,\varphi^h)\|_{\dot B^1_{2,1}})\,\mathrm{d}\tau \tag{5.13}$$

from which and (5.8), we obtain

$$\|(a,\varphi,\mathbf{u})\|^\ell_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|(a_0,\varphi_0,\mathbf{u}_0)\|^\ell_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t \|(f_1,f_2)\|^\ell_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau + \int_0^t \|\mathbf{u}\|_{\dot B^2_{2,1}}(\|(a^\ell,\varphi^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \|(a^h,\varphi^h)\|_{\dot B^1_{2,1}})\,\mathrm{d}\tau. \tag{5.14}$$

In the same manner, we also can get

$$\|(b,\varphi,\mathbf{u})\|^\ell_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|(b_0,\varphi_0,\mathbf{u}_0)\|^\ell_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t \|(f_1,f_2)\|^\ell_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau + \int_0^t \|\mathbf{u}\|_{\dot B^2_{2,1}}(\|(b^\ell,\varphi^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \|(b^h,\varphi^h)\|_{\dot B^1_{2,1}})\,\mathrm{d}\tau \tag{5.15}$$

which combines with (5.14) leads to

$$\|(a,b,\varphi,\mathbf{u})\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|(a_0,b_0,\varphi_0,\mathbf{u}_0)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t \|(f_1,f_2)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau + \int_0^t \|\mathbf{u}\|_{\dot B^{2}_{2,1}}(\|(a^\ell,b^\ell,\varphi^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \|(a^h,b^h,\varphi^h)\|_{\dot B^{1}_{2,1}})\,\mathrm{d}\tau. \tag{5.16}$$

To bound the nonlinear terms in $f_1,\ f_2$, we need the following two crucial estimates which can be obtained from Lemma 2.4 directly.

- $$\|fg\|^{\ell}_{\dot B^{-s}_{2,\infty}} \lesssim \|f\|_{\dot B^{1}_{2,1}}\|g\|_{\dot B^{-s}_{2,\infty}}, \quad -1 < s \le 1, \tag{5.17}$$
- $$\|fg\|^{\ell}_{\dot B^{-s}_{2,\infty}} \lesssim \|f\|_{\dot B^{0}_{2,1}}\|g\|_{\dot B^{-s+1}_{2,\infty}}, \quad 0 < s \le 1. \tag{5.18}$$

To simplify the notation, we set

$$\mathscr{D}_\infty(t) \stackrel{\text{def}}{=} \|(a,\mathbf{u},b)\|^{\ell}_{\dot B^{0}_{2,1}} + \|(a,b)\|^{h}_{\dot B^{1}_{2,1}} + \|\mathbf{u}\|^{h}_{\dot B^{0}_{2,1}},$$
$$\mathscr{D}_1(t) \stackrel{\text{def}}{=} \|(\varphi,\mathbf{u})\|^{\ell}_{\dot B^{2}_{2,1}} + \|\varphi\|^{h}_{\dot B^{1}_{2,1}} + \|\mathbf{u}\|^{h}_{\dot B^{2}_{2,1}}.$$

From (5.17), one has

$$\begin{aligned}
&\|\mathbf{u}\cdot\nabla\varphi^\ell\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \|\varphi\,\mathrm{div}\,\mathbf{u}^\ell\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}}\\
&\lesssim \|\mathbf{u}^\ell\|_{\dot B^{-\sigma}_{2,\infty}}\|\nabla\varphi^\ell\|_{\dot B^{1}_{2,1}} + \|\mathbf{u}\|^{h}_{\dot B^{1}_{2,1}}\|\nabla\varphi^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \|\varphi^\ell\|_{\dot B^{-\sigma}_{2,\infty}}\|\mathrm{div}\,\mathbf{u}^\ell\|_{\dot B^{1}_{2,1}}\\
&\quad + \|\varphi^h\|_{\dot B^{1}_{2,1}}\|\mathrm{div}\,\mathbf{u}^\ell\|_{\dot B^{-\sigma}_{2,\infty}}\\
&\lesssim \|\varphi^\ell\|_{\dot B^{2}_{2,1}}\|\mathbf{u}^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \|\mathbf{u}\|^{h}_{\dot B^{2}_{2,1}}\|\varphi^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \|\mathbf{u}\|^{\ell}_{\dot B^{2}_{2,1}}\|\varphi\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \|\varphi\|^{h}_{\dot B^{1}_{2,1}}\|\mathbf{u}\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}}\\
&\lesssim \mathscr{D}_1(t)\|(\varphi^\ell,\mathbf{u}^\ell)\|_{\dot B^{-\sigma}_{2,\infty}}.
\end{aligned} \tag{5.19}$$

Thanks to (5.18), we have

$$\begin{aligned}
&\|\mathbf{u}\cdot\nabla\varphi^h\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \|\varphi\mathrm{div}\mathbf{u}^h\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}}\\
&\lesssim \|\mathbf{u}\|^{\ell}_{\dot B^{-\sigma+1}_{2,\infty}}\|\nabla\varphi\|^{h}_{\dot B^{0}_{2,1}} + \|\nabla\varphi\|^{h}_{\dot B^{0}_{2,1}}\|\mathbf{u}\|^{h}_{\dot B^{1}_{2,1}} + (\|\varphi\|^{\ell}_{\dot B^{0}_{2,1}} + \|\varphi\|^{h}_{\dot B^{0}_{2,1}})\|\mathrm{div}\mathbf{u}\|^{h}_{\dot B^{1}_{2,1}}\\
&\lesssim \|\varphi\|^{h}_{\dot B^{1}_{2,1}}\|\mathbf{u}\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + (\|\varphi\|^{\ell}_{\dot B^{0}_{2,1}} + \|\varphi\|^{h}_{\dot B^{1}_{2,1}})\|\mathbf{u}\|^{h}_{\dot B^{2}_{2,1}}\\
&\lesssim \mathscr{D}_1(t)\|\mathbf{u}^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + \mathscr{D}_\infty(t)\mathscr{D}_1(t)
\end{aligned} \tag{5.20}$$

which, together with (5.19), gives

$$\|f_1\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \lesssim \mathscr{D}_1(t)\|(\varphi^\ell,\mathbf{u}^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + \mathscr{D}_\infty(t)\mathscr{D}_1(t). \tag{5.21}$$

Next, we bound the terms in f_2. The estimate of $\mathbf{u}\cdot\nabla\mathbf{u}$ follows from essentially the same procedures as $\mathrm{div}(\varphi\mathbf{u})$ so that

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim&\|\mathbf{u}\cdot\nabla\mathbf{u}^{\ell}\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}+\|\mathbf{u}\cdot\nabla\mathbf{u}^{h}\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}\\ \lesssim&\big(\|\mathbf{u}\|^{\ell}_{\dot{B}^{2}_{2,1}}+\|\mathbf{u}\|^{h}_{\dot{B}^{2}_{2,1}}\big)\|\mathbf{u}\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}+\big(\|\mathbf{u}\|^{\ell}_{\dot{B}^{0}_{2,1}}+\|\mathbf{u}\|^{h}_{\dot{B}^{0}_{2,1}}\big)\|\mathbf{u}\|^{h}_{\dot{B}^{2}_{2,1}}\\ \lesssim&\mathscr{D}_1(t)\|\mathbf{u}^{\ell}\|_{\dot{B}^{-\sigma}_{2,\infty}}+\mathscr{D}_\infty(t)\mathscr{D}_1(t).\end{aligned}\tag{5.22}$$

For the term $I(a)(\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u})$, it follows from (5.18) that

$$\begin{aligned}\|I(a)(\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u})\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim&\|\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u}\|_{\dot{B}^{0}_{2,1}}\|I(a)\|_{\dot{B}^{-\sigma+1}_{2,\infty}}\\ \lesssim&\|\mathbf{u}\|_{\dot{B}^{2}_{2,1}}(\|(I(a))^{\ell}\|_{\dot{B}^{-\sigma+1}_{2,\infty}}+\|(I(a))^{h}\|_{\dot{B}^{-\sigma+1}_{2,\infty}})\\ \lesssim&(\|\mathbf{u}^{\ell}\|_{\dot{B}^{2}_{2,1}}+\|\mathbf{u}^{h}\|_{\dot{B}^{2}_{2,1}})(\|(I(a))^{\ell}\|_{\dot{B}^{0}_{2,1}}+\|(I(a))^{h}\|_{\dot{B}^{1}_{2,1}}).\end{aligned}\tag{5.23}$$

In view of the previous estimates (4.18) and (4.19), there holds

$$\|(I(a))^{\ell}\|_{\dot{B}^{0}_{2,1}}+\|(I(a))^{h}\|_{\dot{B}^{1}_{2,1}}\lesssim(1+\mathscr{D}_\infty(t))\mathscr{D}_\infty(t).\tag{5.24}$$

Hence, we infer from (5.23) that

$$\|I(a)(\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u})\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}\lesssim(1+\mathscr{D}_\infty(t))\mathscr{D}_\infty(t)\mathscr{D}_1(t).\tag{5.25}$$

For the last term in f_2, we exploit (5.17) and (5.18) to get

$$\begin{aligned}\|I(a)\nabla\varphi\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim&\|I(a)\|_{\dot{B}^{-\sigma}_{2,\infty}}\|\nabla\varphi^{\ell}\|_{\dot{B}^{1}_{2,1}}+\|I(a)\|_{\dot{B}^{-\sigma+1}_{2,\infty}}\|\nabla\varphi^{h}\|_{\dot{B}^{0}_{2,1}}\\ \lesssim&\|I(a)\|_{\dot{B}^{-\sigma}_{2,\infty}}\|\varphi^{\ell}\|_{\dot{B}^{2}_{2,1}}+\|I(a)\|_{\dot{B}^{-\sigma+1}_{2,\infty}}\|\varphi^{h}\|_{\dot{B}^{1}_{2,1}}.\end{aligned}\tag{5.26}$$

Keeping in mind that $I(a)=a-aI(a)$, we use Lemma 2.4 and (4.18), (4.19) to write

$$\begin{aligned}\|I(a)\|_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim&\|a\|_{\dot{B}^{-\sigma}_{2,\infty}}+\|aI(a)\|_{\dot{B}^{-\sigma}_{2,\infty}}\\ \lesssim&\|a\|_{\dot{B}^{-\sigma}_{2,\infty}}+\|a\|_{\dot{B}^{-\sigma}_{2,\infty}}\|I(a)\|_{\dot{B}^{1}_{2,1}}\\ \lesssim&(1+\|(I(a))^{\ell}\|_{\dot{B}^{0}_{2,1}}+\|(I(a))^{h}\|_{\dot{B}^{1}_{2,1}})(\|a^{\ell}\|_{\dot{B}^{-\sigma}_{2,\infty}}+\|a^{h}\|_{\dot{B}^{1}_{2,1}})\\ \lesssim&(1+(\mathscr{D}_\infty(t))^2)\|a^{\ell}\|_{\dot{B}^{-\sigma}_{2,\infty}}+(1+(\mathscr{D}_\infty(t))^2)\mathscr{D}_\infty(t).\end{aligned}\tag{5.27}$$

Similar to previous estimate, one has

$$\|I(a)\|_{\dot{B}^{-\sigma+1}_{2,\infty}}\lesssim\|(I(a))^{\ell}\|_{\dot{B}^{0}_{2,1}}+\|(I(a))^{h}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\lesssim(1+\mathscr{D}_\infty(t))\mathscr{D}_\infty(t).\tag{5.28}$$

Taking (5.27) and (5.28) into (5.26) gives

$$\|I(a)\nabla\varphi\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \lesssim (1+(\mathscr{D}_\infty(t))^2)\mathscr{D}_1(t)\|a^\ell\|_{\dot B^{-\sigma}_{2,\infty}} + (1+(\mathscr{D}_\infty(t))^2)\mathscr{D}_\infty(t)\mathscr{D}_1(t). \tag{5.29}$$

Collecting the estimates (5.22), (5.25), and (5.29), we obtain

$$\|f_2\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \lesssim (1+(\mathscr{D}_\infty(t))^2)\mathscr{D}_1(t)\|(a^\ell,\mathbf{u}^\ell)\|_{\dot B^{-\sigma}_{2,\infty}} + (1+\mathscr{D}_\infty(t))\mathscr{D}_\infty(t)\mathscr{D}_1(t). \tag{5.30}$$

Plugging (5.21) and (5.30) into (5.16), we finally arrive at

$$\begin{aligned}\|(a,b,\varphi,\mathbf{u})\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \lesssim & \|(a_0,b_0,\varphi_0,\mathbf{u}_0)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t (1+\mathscr{D}_\infty(\tau))\mathscr{D}_\infty(\tau)\mathscr{D}_1(\tau)\,\mathrm{d}\tau \\ & + \int_0^t (1+(\mathscr{D}_\infty(\tau))^2)\mathscr{D}_1(\tau)\|(a,b,\varphi,\mathbf{u})\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}}\,\mathrm{d}\tau. \end{aligned}\tag{5.31}$$

Noticing that the definition of φ_0 in (4.2), it is easy to deduce from Lemma 2.4 that

$$\begin{aligned}\|\varphi_0\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \lesssim & \|(a_0,b_0)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \|(a_0,b_0)\|_{\dot B^{-\sigma}_{2,\infty}}\|(a_0,b_0)\|_{\dot B^{1}_{2,1}} \\ \lesssim & \|(a_0,b_0)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + (\|(a_0,b_0)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} + \|(a_0^h,b_0^h)\|_{\dot B^{1}_{2,1}})(\|(a_0^\ell,b_0^\ell)\|_{\dot B^{0}_{2,1}} \\ & + \|(a_0^h,b_0^h)\|_{\dot B^{1}_{2,1}}).\end{aligned}$$

Consequently, one can employ nonlinear generalizations of the Gronwall's inequality to get

$$\|(a,b,\mathbf{u},\varphi)(t,\cdot)\|^{\ell}_{\dot B^{-\sigma}_{2,\infty}} \le C_0 \tag{5.32}$$

for all $t \ge 0$, where $C_0 > 0$ depends on the norm of $a_0, b_0, \mathbf{u}_0$. This completes the proof of Proposition 5.1. □

Now, we prove the Lyapunov-type inequality from (5.3). For any $0 < \sigma \le 1$, it follows from an interpolation inequality that

$$\|(\varphi,\mathbf{u})\|^{\ell}_{\dot B^{0}_{2,1}} \le C\big(\|(\varphi,\mathbf{u})\|_{\dot B^{-\sigma}_{2,\infty}}\big)^{\alpha_1}\big(\|(\varphi,\mathbf{u})\|^{\ell}_{\dot B^{2}_{2,1}}\big)^{1-\alpha_1}, \quad \alpha_1 = \frac{2}{2+\sigma} \in (0,1),$$

which, together with Proposition 5.1, implies

$$\|(\varphi,\mathbf{u})\|^{\ell}_{\dot B^{2}_{2,1}} \ge c_0\big(\|(\varphi,\mathbf{u})\|^{\ell}_{\dot B^{0}_{2,1}}\big)^{\frac{1}{1-\alpha_1}}. \tag{5.33}$$

Taking $\beta = 1+\alpha_1 > 0$ in (5.4) and (5.5) and combining with (5.33), we deduce from (5.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^0_{2,1}} + \|\varphi\|^{h}_{\dot{B}^1_{2,1}} + \|\mathbf{u}\|^{h}_{\dot{B}^0_{2,1}}) + \widetilde{c}_0(\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^0_{2,1}} + \|\varphi\|^{h}_{\dot{B}^1_{2,1}} + \|\mathbf{u}\|^{h}_{\dot{B}^0_{2,1}})^{1+\frac{2}{\sigma}} \le 0. \tag{5.34}$$

Solving this differential inequality directly, we obtain

$$\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^0_{2,1}} + \|\varphi\|^{h}_{\dot{B}^1_{2,1}} + \|\mathbf{u}\|^{h}_{\dot{B}^0_{2,1}} \le C(1+t)^{-\frac{\sigma}{2}}. \tag{5.35}$$

For any $-\sigma < \gamma_1 < 0$, by the interpolation inequality, we have

$$\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^{\gamma_1}_{2,1}} \le C\big(\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^{-\sigma}_{2,\infty}}\big)^{\alpha_2}\big(\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^0_{2,1}}\big)^{1-\alpha_2}, \quad \alpha_2 = -\frac{\gamma_1}{\sigma} \in (0, 1),$$

which, together with Proposition 5.1, gives

$$\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^{\gamma_1}_{2,1}} \le C(1+t)^{\frac{-\sigma(1-\alpha_2)}{2}} = C(1+t)^{-\frac{\gamma_1+\sigma}{2}}. \tag{5.36}$$

In the light of $-\sigma < \gamma_1 < 0$, we see that

$$\|(\varphi^h, \mathbf{u}^h)\|_{\dot{B}^{\gamma_1}_{2,1}} \le C(\|\varphi\|^{h}_{\dot{B}^1_{2,1}} + \|\mathbf{u}\|^{h}_{\dot{B}^0_{2,1}}) \le C(1+t)^{-\frac{\sigma}{2}},$$

which, together with (5.36), yields

$$\begin{aligned}\|(\varphi, \mathbf{u})\|_{\dot{B}^{\gamma_1}_{2,1}} &\le C(\|(\varphi, \mathbf{u})\|^{\ell}_{\dot{B}^{\gamma_1}_{2,1}} + \|(\varphi, \mathbf{u})\|^{h}_{\dot{B}^{\gamma_1}_{2,1}}) \\ &\le C(1+t)^{-\frac{\gamma_1+\sigma}{2}} + C(1+t)^{-\frac{\sigma}{2}} \\ &\le C(1+t)^{-\frac{\gamma_1+\sigma}{2}}.\end{aligned}$$

Hence, thanks to the embedding relation $\dot{B}^0_{2,1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, one infers that

$$\|\Lambda^{\gamma_1}(\varphi, \mathbf{u})\|_{L^2} \le C(1+t)^{-\frac{\gamma_1+\sigma}{2}}.$$

This completes the proof of Theorem 1.3. □

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