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GLOBAL EXISTENCE AND STABILITY FOR THE 2D OLDROYD-B MODEL WITH MIXED PARTIAL DISSIPATION

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ABSTRACT. This paper focuses on a two-dimensional incompressible Oldroyd-B model with mixed partial dissipation. The goal here is to establish the small data global existence and stability in the Sobolev space $H^2(\mathbb{R}^2)$. The velocity equation itself, without coupling with the equation of the non-Newtonian stress tensor, is an anisotropic 2D Navier-Stokes whose solutions are not known to be stable in Sobolev spaces due to potential rapid growth in time. By unearthing the hidden wave structure of the system and exploring the smoothing and stabilizing effect of the non-Newtonian stress tensor on the fluid, we are able to solve the desired global existence and stability problem.

1. Introduction

A class of models of complex fluids is based on an equation for a solvent coupled with a kinetic description of particles suspended in it. In the case of dilute suspensions weakly confined by a Hookean spring potential, a rigorously established exact closure for the moments in the kinetic equation of this Navier–Stokes–Fokker–Planck system yields the Oldroyd-B system (see, e.g., [2, 8, 31]). The standard Oldroyd-B model can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \mu_1 \nabla \cdot \tau, \\ \partial_t \tau + u \cdot \nabla \tau + Q(\tau, \nabla u) + a\tau = \eta \Delta \tau + \mu_2 D(u), \\ \nabla \cdot u = 0, \end{cases}$$

where $u = u(x,t)$ represents the velocity field of the fluid, $p = p(x,t)$ the pressure and $\tau = \tau(x,t)$ (a symmetric matrix) the non-Newtonian added stress tensor, and ν, μ_1, a, η and μ_2 are nonnegative real parameters. Here $D(u)$ is the symmetric part of the velocity gradient defined by

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

The bilinear term Q reads

$$Q(\tau, \nabla u) = \tau W(u) - W(u)\tau - b(D(u)\tau + \tau D(u)),$$

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where $b \in [-1, 1]$ is a parameter and $W(u)$ is the skew-symmetric part of the ∇u,

$$W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

Fundamental issues such as the global existence and the stability problems on the Oldroyd-B models have recently attracted considerable interests. There are substantial developments and significant progress has been made. Interested readers may consult the references listed here (see, e.g., [1, 4–15, 17–19, 21, 22, 24, 25, 27, 29, 30, 36–39, 41–45]). Understandably this list represents only a small portion of the large literature on this subject. This paper focuses on the following anisotropic Oldroyd-B system

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \partial_{11} u + \nabla \cdot \tau, \quad x \in \mathbb{R}^2, t > 0, \\ \partial_t \tau + u \cdot \nabla \tau + Q(\tau, \nabla u) + \tau = \partial_{22} \tau + D(u), \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

which involves only horizontal kinematic dissipation and vertical dissipation in the equation of τ. (1.1) may be relevant for certain anisotropic complex fluids. The anisotropic Navier-Stokes equations have been used in the modeling of many fluids such as turbulent flows in Ekman layers [32]. The equation of τ can be derived from the equation of the conformation tensor by replacing the damping term related to the Weissenberg number by a dissipative differential operator term [11]. It has become a common practice in the modeling and numerical simulations of viscoelastic fluids to add stress diffusion (sometimes anisotropic stress diffusion) in order to effectively stabilize the stress and the numerical calculations. The effect of stress diffusion on the dynamics of creeping viscoelastic flow has been analyzed (see, e.g., [20, 26, 33, 35]). The study of this paper would help fill the gap on how the anisotropic stress tensor would affect the dynamics of viscoelastic flow. The goal of this paper is to solve the small data global existence and stability problem. Without loss of generality, we have set the parameters in (1.1) equal to 1 for notational convenience.

The lack of vertical velocity dissipation makes the stability problem concerned here difficult. The corresponding vorticity $\omega = \nabla \times u$ satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \partial_{11} \omega + \nabla \times \nabla \cdot \tau, \qquad x \in \mathbb{R}^2,\ t > 0 \tag{1.2}$$

and it does not appear possible to establish any uniform-in-time bound on the Sobolev norms of ω. Even when $\tau = 0$, the vorticity gradient $\nabla \omega$ for the anisotropic 2D Navier-Stokes equation

$$\partial_t \omega + u \cdot \nabla \omega = \partial_{11} \omega, \qquad x \in \mathbb{R}^2,\ t > 0 \tag{1.3}$$

may grow in time. In fact, the only upper bound on $\nabla \omega$ for (1.3) is double exponential in time, for any $2 \le q \le \infty$,

$$\|\nabla \omega(t)\|_{L^q} \le (\|\nabla \omega_0\|_{L^q})^{e^{C\, \|\omega_0\|_{L^\infty}\, t}}.$$

The double exponential growth rate was confirmed for the 2D Euler equation in a unit disk by Kiselev and Šverák [23]. The growth rate for the 2D Euler equation on a more general smooth bounded domain was explored by Xu [40]. Whether the double exponential upper bound for the 2D Euler or for the anisotropic Navier-Stokes in the whole space $\mathbb{R}^2$ is sharp remains an open problem.

In the case when the 2D Oldroyd-B model has both damping and full Laplacian dissipation in the equation of τ, Elgindi and Rousset [13] were able to overcome the difficulty by considering a combined quantity $G := \omega - \nabla \times \nabla \cdot \Delta^{-1}\tau$ and its equation, and successfully solved the small data global well-posedness problem. The 3D Oldroyd-B model has both damping and full Laplacian dissipation was dealt with by Elgindi and Liu [12]. The damping term in the equation of τ plays a crucial role in the approaches of [12, 13].

Very recently Constantin, Wu, Zhao and Zhu [11] considered the d-dimensional ($d = 2, 3$) Oldroyd-B model with only fractional dissipation $(-\Delta)^{\beta}\tau$ and without damping in τ. [11] derived a system of special wave equations satisfied by u and $\mathbb{P}\nabla \cdot \tau$, where

$$\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$$

denotes the Leray projection operator. As a consequence, [11] observed that the non-Newtonian stress has a stabilizing effect on the fluid and was able to establish the small data global well-posedness and stability for any $\beta \geq \frac{1}{2}$.

Our Oldroyd-B model in (1.1) also admits a wave structure. By applying the Leray projection operator $\mathbb{P}$ to eliminate the pressure term, we obtain

$$\partial_t u = \partial_{11} u + \mathbb{P}(\nabla \cdot \tau) + N_1, \qquad N_1 = \mathbb{P}(-u \cdot \nabla u). \tag{1.4}$$

Applying $\mathbb{P}\nabla\cdot$ to the equation of τ, we have

$$\partial_t \mathbb{P}\nabla \cdot \tau = \partial_{22}\mathbb{P}\nabla \cdot \tau - \mathbb{P}\nabla \cdot \tau + \frac{1}{2}\Delta u + N_2 \tag{1.5}$$

with

$$N_2 = -\mathbb{P}\nabla \cdot (u \cdot \nabla\tau) - \mathbb{P}\nabla \cdot Q(\tau, \nabla u).$$

Differentiating (1.4) and (1.5) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt} u + (1-\Delta)\partial_t u - \partial_{11}(1-\partial_{22})u - \frac{1}{2}\Delta u = N_3, \\ \partial_{tt}\mathbb{P}(\nabla \cdot \tau) + (1-\Delta)\partial_t \mathbb{P}(\nabla \cdot \tau) - \partial_{11}(1-\partial_{22})\mathbb{P}(\nabla \cdot \tau) - \frac{1}{2}\Delta\mathbb{P}(\nabla \cdot \tau) = N_4, \end{cases} \tag{1.6}$$

where N_3 and N_4 are given by

$$N_3 = (\partial_t + 1)N_1 + N_2, \qquad N_4 = (\partial_t - \partial_{11})N_2 + \frac{1}{2}\Delta N_1.$$

The wave structure derived above is a consequence of the coupling between the equations of u and τ. Without the coupling and even for $\tau = 0$, the linearized equation of u is given by

$$\partial_t u = \partial_{11} u. \tag{1.7}$$

Clearly the linearized wave equation for u given by

$$\partial_{tt} u + (1-\Delta)\partial_t u - \partial_{11}(1-\partial_{22})u - \frac{1}{2}\Delta u = 0 \tag{1.8}$$

is much more regularized than (1.7). We shall exploit the wave structure in (1.6) to gain extra regularization and damping properties. One crucial regularity to be extracted is the time integrability of the derivatives of u, not just the horizontal derivatives. This is a consequence of the full Laplacian operator in (1.8). When we seek a solution (u, τ) of (1.1) in the Sobolev space H^2, we expect to gain the uniform time integrability, for a constant $C > 0$ and for any $t > 0$,

$$\int_0^t \|\nabla u(s)\|_{H^1}^2 \, ds \leq C < \infty. \tag{1.9}$$

Besides understanding the time integrability in (1.9) from the wave structure, there is another simple way to comprehend (1.9). It is really the coupling in (1.4) and (1.5) that allows us to transfer the time integrability from one function in the system to another. More precisely, we can represent Δu in terms of the rest in (1.5),

$$\Delta u = 2\partial_t \mathbb{P}\nabla\cdot\tau - 2\partial_{22}\mathbb{P}\nabla\cdot\tau + 2\mathbb{P}\nabla\cdot\tau - 2N_2 \tag{1.10}$$

then

$$\begin{aligned}
\|\nabla u\|_{H^1}^2 &= -(u,\Delta u) - (\nabla u, \nabla\Delta u)\\
&= -2\int u\cdot\partial_t\mathbb{P}\nabla\cdot\tau + 2\int u\cdot\partial_{22}\mathbb{P}\nabla\cdot\tau\,dx\\
&\quad - 2\int u\cdot\mathbb{P}\nabla\cdot\tau\,dx + 2\int u\cdot N_2\,dx\\
&\quad - 2\int \nabla u\cdot\nabla\partial_t\mathbb{P}\nabla\cdot\tau + 2\int\nabla u\cdot\nabla\partial_{22}\mathbb{P}\nabla\cdot\tau\,dx\\
&\quad - 2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau\,dx + 2\int\nabla u\cdot\nabla N_2\,dx,
\end{aligned}$$

where (f,g) above denotes the L^2-inner product. The time integrability of $\|\nabla u\|_{H^1}^2$ is then converted to the time integrability of other terms. This explains our strategy on how to make use of the stabilizing effect of τ on the fluid to prevent the growth of the Sobolev norms of the velocity. We are now ready to state our main result.

Theorem 1.1. *Assume the initial data $(u_0,\tau_0)\in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0 = 0$. Then, there exists a constant $\varepsilon>0$ such that, if*

$$\|u_0\|_{H^2} + \|\tau_0\|_{H^2} \le \varepsilon,$$

then (1.1) has a unique global classical solution (u,τ) satisfying, for any $t>0$,

$$(\|u\|_{H^2}^2 + \|\tau\|_{H^2}^2) + 2\int_0^t (\|\partial_1 u(s)\|_{H^2}^2 + \|\partial_2\tau(s)\|_{H^2}^2 + \|\tau(s)\|_{H^2}^2 + \|\nabla u(s)\|_{H^1}^2)\,ds \le C\,\varepsilon^2,$$

where $C>0$ is pure constant.

We make two remarks about Theorem 1.1.

Remark 1.2.

(1) The damping term in τ appears to be necessary in order to bound Q in the L^2-estimate. Q generates a term of the form $\|\tau\|_{L^2}^2$, which requires damping in τ to yield a suitable upper bound.
(2) When the combination of $\partial_{11}u$ and $\partial_{22}\tau$ is replaced by that of $\partial_{22}u$ and $\partial_{11}\tau$, Theorem 1.1 remains valid. We just need to slightly modify the proof. Therefore, as long as the dissipation of u and τ are in different directions, the nonlinear terms can be bounded suitably and the result still holds. Physically the dissipation of u and τ in different directions helps complement the regularization of each other, and thus controls the nonlinearity.

The local-in-time existence and uniqueness of solutions to (1.1) can be shown via standard approaches such as those in the book of Majda and Bertozzi [28]. Our focus will be on the global-in-time bound of (u,τ) in H^2. One of the most suitable methods for this purpose is the bootstrapping argument (see, e.g., [34, p.21]). To proceed, we first define a suitable energy functional

$$E(t) = E_1(t) + E_2(t),$$

with

$$E_1(t) := \sup_{0\le s\le t} (\|u\|_{H^2}^2 + \|\tau\|_{H^2}^2) + 2\int_0^t (\|\partial_1 u(s)\|_{H^2}^2 + \|\partial_2 \tau(s)\|_{H^2}^2 + \|\tau(s)\|_{H^2}^2)\, ds,$$

$$E_2(t) := \int_0^t \|\nabla u(s)\|_{H^1}^2 \, ds.$$

E_1 represents the standard energy consisting of the H^2-norm of (u, τ) and the associated time integrals parts from the horizontal dissipation in u and the vertical dissipation and damping in τ. E_2 is the time integral in (1.9) representing the extra regularization through the coupling. Our main efforts are devoted to proving that, for any $t > 0$,

$$E(t) \le C_1 E(0) + C_2 E^{\frac{3}{2}}(t). \tag{1.11}$$

The bootstrapping argument applied to (1.11) then implies that, if $E(0) \le \varepsilon^2$ for some suitable $\varepsilon > 0$, then, for a constant $C > 0$ and any $t > 0$,

$$E(t) \le C\,\varepsilon^2,$$

which, in particular, asserts the desired global bound on the H^2-norm of (u, τ). The details are provided in Section 2.

2. Proof of Theorem 1.1

This section details the proof of Theorem 1.1. First we list several anisotropic inequalities to be used frequently in the proof.

The first is an anisotropic upper bound for a triple product, a very useful tool in bounding the nonlinearity when the dissipation is anisotropic. Its proof can be found in [3].

Lemma 2.1. *Assume that f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then,*

$$\left|\int_{\mathbb{R}^2} fgh \, dx\right| \le 2^{\frac{3}{2}} \|f\|_{L^2} \|g\|_{L^2}^{\frac12} \|\partial_2 g\|_{L^2}^{\frac12} \|h\|_{L^2}^{\frac12} \|\partial_1 h\|_{L^2}^{\frac12}.$$

The second lemma provides an upper bound for the L^∞-norm of a 2D function in terms of the H^1-norm of its horizontal or vertical derivatives.

Lemma 2.2. *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac14} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac14} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac14} \|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac14}.$$

Consequently,

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac12} \|\partial_1 f\|_{H^1}^{\frac12}, \quad \|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac12} \|\partial_2 f\|_{H^1}^{\frac12}.$$

The proof of Lemma 2.2 can be found in [16].

Proof. As explained in the introduction, it suffices to prove (1.11). For the sake of clarity, we prove the following two inequalities, one for E_1 and one for E_2,

$$E_1 \le E(0) + C_1\, E_1^{\frac{3}{2}}(t) + C_2\, E_2^{\frac{3}{2}}(t), \tag{2.1}$$

$$E_2 \le C_3 E(0) + C_4 E_1(t) + C_5 E_1^{\frac{3}{2}}(t) + C_6 E_2^{\frac{3}{2}}(t), \tag{2.2}$$

where C_1 through C_6 are positive pure constants. Then $E_1 + \frac{1}{2C_4}E_2$ yields

$$\begin{aligned} E_1 + \frac{1}{2C_4}E_2 &\le E(0) + C_1\, E_1^{\frac{3}{2}}(t) + C_2\, E_2^{\frac{3}{2}}(t) \\ &\quad + \frac{C_3}{2C_4}E(0) + \frac{1}{2}E_1(t) + \frac{C_5}{2C_4}E_1^{\frac{3}{2}}(t) + \frac{C_6}{2C_4}E_2^{\frac{3}{2}}(t) \end{aligned}$$

or

$$\frac{1}{2}E_1 + \frac{1}{2C_4}E_2 \le \left(1 + \frac{C_3}{2C_4}\right)E(0) + \left(C_1 + \frac{C_5}{2C_4}\right)E_1^{\frac{3}{2}}(t) + \left(C_2 + \frac{C_6}{2C_4}\right)E_2^{\frac{3}{2}}(t)$$

or

$$E(t) \le \widetilde{C}_1 E(0) + \widetilde{C}_2 E^{\frac{3}{2}}(t). \tag{2.3}$$

We take the initial data (u_0, τ_0) to be sufficiently small, say

$$E(0) = \|(u_0, \tau_0)\|_{H^2}^2 \le \frac{1}{16\widetilde{C}_1\, \widetilde{C}_2^2} := \varepsilon^2.$$

Then the bootstrapping argument applied to (2.3) yields, for all

$$E(t) \le \frac{1}{8\widetilde{C}_2^2} := 2\widetilde{C}_1\varepsilon^2.$$

In fact, if we make the ansatz that

$$E(t) \le \frac{1}{4\widetilde{C}_2^2}, \tag{2.4}$$

then (2.3) implies

$$E(t) \le \widetilde{C}_1 E(0) + \widetilde{C}_2 \frac{1}{2\widetilde{C}_2}\, E(t) \quad \text{or} \quad \frac{1}{2}E(t) \le \widetilde{C}_1 E(0)$$

or

$$E(t) \le \frac{1}{8\widetilde{C}_2^2},$$

which is half of the bound in the ansatz (2.4). The bootstrapping argument then asserts that this bound actually holds for all $t > 0$. This yields the desired global uniform bound on $\|(u(t), \tau(t))\|_{H^2}$.

It remains to prove (2.1) and (2.2). We first prove (2.1). Due to the equivalence

$$\|f\|_{H^2} \quad \sim \quad \|f\|_{L^2} + \|\Delta f\|_{L^2}, \tag{2.5}$$

we just need to bound $\|(u,\tau)\|_{L^2}$ and $\|(\Delta u, \Delta\tau)\|_{L^2}$. Dotting (1.1) by (u,τ), and applying Δ to (1.1) and dotting the resulting equation by $(\Delta u, \Delta\tau)$, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|(u,\tau)\|_{L^2}^2 + \|(\Delta u, \Delta\tau)\|_{L^2}^2) \\ &+ \|\partial_1 u\|_{L^2}^2 + \|\partial_1\Delta u\|_{L^2}^2 + \|\partial_2\tau\|_{L^2}^2 + \|\Delta\partial_2\tau\|_{L^2}^2 + \|\tau\|_{L^2}^2 + \|\Delta\tau\|_{L^2}^2 \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
I_1 &= -(\Delta(u\cdot\nabla u),\Delta u),\\
I_2 &= -(\Delta(u\cdot\nabla\tau),\Delta\tau),\\
I_3 &= -(Q(\tau,\nabla u),\tau)-(\Delta Q(\tau,\nabla u),\Delta\tau).
\end{aligned}$$

Here we have used the facts, due to $\nabla\cdot u=0$ and $\tau_{ij}=\tau_{ji}$ for $i,j=1,2$,

$$\begin{aligned}
&\int u\cdot(u\cdot\nabla u)\,dx=0,\quad \int\tau\cdot(u\cdot\nabla\tau)\,dx=0,\\
&\int(u\cdot(\nabla\cdot\tau)+D(u)\cdot\tau)\,dx=0,\quad \int(\Delta u\cdot\Delta(\nabla\cdot\tau)+\Delta D(u)\cdot\Delta\tau)\,dx=0.
\end{aligned}$$

We now bound I_1. By $\nabla\cdot u=0$ and Lemma 2.1,

$$\begin{aligned}
I_1 &= -\int\Delta u\cdot(\Delta u\cdot\nabla u)\,dx-2\int\Delta u\cdot(\nabla u\cdot\nabla^2u)\,dx\\
&\le C\,\|\Delta u\|_{L^2}\,\|\Delta u\|_{L^2}^{\frac12}\,\|\partial_1\Delta u\|_{L^2}^{\frac12}\,\|\nabla u\|_{L^2}^{\frac12}\,\|\partial_2\nabla u\|_{L^2}^{\frac12}\\
&\quad +C\,\|\Delta u\|_{L^2}\,\|\nabla^2u\|_{L^2}^{\frac12}\,\|\partial_1\nabla^2u\|_{L^2}^{\frac12}\,\|\nabla u\|_{L^2}^{\frac12}\,\|\partial_2\nabla u\|_{L^2}^{\frac12}\\
&\le C\,\|u\|_{H^2}\,\|\nabla u\|_{H^1}^{\frac32}\,\|\partial_1 u\|_{H^2}^{\frac12}\\
&\le C\,\|u\|_{H^2}\,(\|\nabla u\|_{H^1}^2+\|\partial_1u\|_{H^2}^2).
\end{aligned}$$

By $\nabla\cdot u=0$ and Lemma 2.1,

$$\begin{aligned}
I_2 &= -\int\Delta\tau\cdot(\Delta u\cdot\nabla\tau)\,dx-2\int\Delta\tau\cdot(\nabla u\cdot\nabla^2\tau)\,dx\\
&\le C\,\|\Delta\tau\|_{L^2}\,\|\Delta u\|_{L^2}^{\frac12}\,\|\partial_1\Delta u\|_{L^2}^{\frac12}\,\|\nabla\tau\|_{L^2}^{\frac12}\,\|\partial_2\nabla\tau\|_{L^2}^{\frac12}\\
&\quad +C\,\|\Delta\tau\|_{L^2}\,\|\nabla u\|_{L^2}^{\frac12}\,\|\partial_1\nabla u\|_{L^2}^{\frac12}\,\|\nabla^2\tau\|_{L^2}^{\frac12}\,\|\partial_2\nabla^2\tau\|_{L^2}^{\frac12}\\
&\le C\,(\|u\|_{H^2}+\|\tau\|_{H^2})\,(\|\tau\|_{H^2}^2+\|\partial_1u\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2).
\end{aligned}$$

Naturally I_3 is divided into two parts $I_3=I_{3,1}+I_{3,2}$ with

$$I_{3,1}=-(Q(\tau,\nabla u),\tau),\qquad I_{3,2}=(\Delta Q(\tau,\nabla u),\Delta\tau).$$

By Lemma 2.1,

$$I_{3,1}\le C\|\tau\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|\tau\|_{L^2}^{\frac12}\|\partial_2\tau\|_{L^2}^{\frac12}\le C\|u\|_{H^2}\|\tau\|_{H^2}^2.$$

To distinguish between the horizontal and the vertical derivatives, we rewrite $I_{3,2}$ as

$$\begin{aligned}
I_{3,2} &= -\int\Delta Q\cdot\Delta\tau\;dx=-\int(\partial_{11}Q+\partial_{22}Q)\cdot(\partial_{11}\tau+\partial_{22}\tau)\;dx\\
&= -\int(\partial_{11}Q\cdot\partial_{11}\tau+\partial_{11}Q\cdot\partial_{22}\tau+\partial_{22}Q\cdot\partial_{11}\tau+\partial_{22}Q\cdot\partial_{22}\tau)\;dx\\
&= I_{3,2,1}+I_{3,2,2}+I_{3,2,3}+I_{3,2,4}.
\end{aligned}$$

By Hölder's inequality and Lemma 2.2,

$$\begin{aligned}
I_{3,2,1} &= -\int \partial_{11}\tau\cdot\nabla u\cdot\partial_{11}\tau + 2\partial_1\tau\cdot\partial_1\nabla u\cdot\partial_{11}\tau + \tau\cdot\partial_{11}\nabla u\cdot\partial_{11}\tau \; dx\\
&\le C\|\partial_{11}\tau\|_{L^2}\|\nabla u\|_{L^\infty}\|\partial_{11}\tau\|_{L^2} + C\|\partial_1\tau\|_{L^\infty}\|\partial_1\nabla u\|_{L^2}\|\partial_{11}\tau\|_{L^2}\\
&\quad + C\|\tau\|_{L^\infty}\|\partial_{11}\nabla u\|_{L^2}\|\partial_{11}\tau\|_{L^2}\\
&\le C\|\tau\|_{H^2}^2\|\nabla u\|_{H^1}^{\frac12}\|\partial_1\nabla u\|_{H^1}^{\frac12} + C\|\partial_1\tau\|_{H^1}^{\frac12}\|\partial_2\partial_1\tau\|_{H^1}^{\frac12}\|\partial_1\nabla u\|_{L^2}\|\tau\|_{H^2}\\
&\quad + C\|\partial_1 u\|_{H^2}\|\tau\|_{H^2}^2\\
&\le C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2).
\end{aligned}$$

By integration by parts and Lemma 2.2,

$$\begin{aligned}
I_{3,2,2} &= \int \partial_1 Q\cdot\partial_{122}\tau \; dx\\
&\le \|\tau\|_{L^\infty}\|\partial_1\nabla u\|_{L^2}\|\partial_2\partial_{12}\tau\|_{L^2} + \|\partial_1\tau\|_{L^2}\|\nabla u\|_{L^\infty}\|\partial_2\partial_{12}\tau\|_{L^2}\\
&\le \|\partial_2\tau\|_{H^2}\|\tau\|_{H^2}\|u\|_{H^2} + \|\tau\|_{H^2}\|\partial_1 u\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|\partial_2\tau\|_{H^2}\\
&\le C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2).
\end{aligned}$$

$I_{3,2,3}$ has the same bound as $I_{3,2,2}$. The estimate for $I_{3,2,4}$ is also similar,

$$\begin{aligned}
I_{3,2,4} &= \int \partial_2 Q\cdot\partial_{222}\tau \; dx\\
&\le C\|\tau\|_{L^\infty}\|\partial_2\nabla u\|_{L^2}\|\partial_{222}\tau\|_{L^2} + \|\partial_2\tau\|_{L^2}\|\nabla u\|_{L^\infty}\|\partial_{222}\tau\|_{L^2}\\
&\le C\|\partial_2\tau\|_{H^2}\|\tau\|_{H^2}\|u\|_{H^2} + \|\partial_2\tau\|_{H^2}^{\frac32}\|\tau\|_{H^2}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\\
&\le C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2).
\end{aligned}$$

Combining the bounds above leads to

$$\begin{aligned}
I_3 = I_{3,1}+I_{3,2} &\le C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2),\\
I_1+I_2+I_3 &\le C\,\|u\|_{H^2}\,(\|\nabla u\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2)\\
&\qquad + C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2).
\end{aligned}$$

Inserting the upper bound for $I_1+I_2+I_3$ in (2.6), integrating in time and invoking the norm equivalence (2.5), we find

$$\begin{aligned}
&\|(u,\tau)\|_{H^2}^2 + 2\int_0^t (\|\partial_1 u\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2+\|\tau\|_{H^2}^2)\,ds\\
&\le \|(u_0,\tau_0)\|_{H^2}^2 + C\int_0^t \|u\|_{H^2}\,(\|\nabla u\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2)\,ds\\
&\quad + C\int_0^t (\|u\|_{H^2}+\|\tau\|_{H^2})(\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2)\,ds\\
&\le \|(u_0,\tau_0)\|_{H^2}^2 + C\sup_{0\le s\le t}\|u(s)\|_{H^2}\int_0^t (\|\nabla u\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2)\,ds\\
&\quad + C\sup_{0\le s\le t}(\|u\|_{H^2}+\|\tau\|_{H^2})\int_0^t (\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2)\,ds\\
&\le E(0) + C\,E_1^{\frac32}(t) + C\,E_2^{\frac32}(t).
\end{aligned}$$

This proves (2.1), namely

$$E_1(t) \le E(0) + C\, E_1^{\frac{3}{2}}(t) + C\, E_2^{\frac{3}{2}}(t).$$

To prove (2.2), we invoke (1.5) or (1.10) to write $\|\nabla u\|_{H^1}^2$ as

$$\begin{aligned}
\|\nabla u\|_{H^1}^2 &= -(u, \Delta u) - (\nabla u, \nabla \Delta u) \\
&= -2\int u \cdot \partial_t \mathbb{P}\nabla \cdot \tau\, dx + 2\int u \cdot \partial_{22}\mathbb{P}\nabla \cdot \tau\, dx \\
&\quad - 2\int u \cdot \mathbb{P}\nabla \cdot \tau\, dx + 2\int u \cdot N_2\, dx \\
&\quad - 2\int \nabla u \cdot \nabla \partial_t \mathbb{P}\nabla \cdot \tau\, dx + 2\int \nabla u \cdot \nabla \partial_{22}\mathbb{P}\nabla \cdot \tau\, dx \\
&\quad - 2\int \nabla u \cdot \nabla \mathbb{P}\nabla \cdot \tau\, dx + 2\int \nabla u \cdot \nabla N_2\, dx,
\end{aligned} \tag{2.7}$$

where

$$N_2 = -\mathbb{P}\nabla \cdot (u \cdot \nabla \tau) - \mathbb{P}\nabla \cdot Q(\tau, \nabla u).$$

In addition,

$$\begin{aligned}
\int u \cdot \partial_t \mathbb{P}\nabla \cdot \tau\, dx &= \frac{d}{dt}\int u \cdot \mathbb{P}\nabla \cdot \tau dx - \int \mathbb{P}\nabla \cdot \tau \cdot \partial_t u\, dx \\
&= \frac{d}{dt}\int u \cdot \mathbb{P}\nabla \cdot \tau dx \\
&\quad - \int \mathbb{P}\nabla \cdot \tau \cdot (\partial_{11} u + \mathbb{P}(\nabla \cdot \tau) + \mathbb{P}(-u \cdot \nabla u))\, dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int \nabla u \cdot \partial_t \nabla \mathbb{P}\nabla \cdot \tau\, dx &= \frac{d}{dt}\int \nabla u \cdot \nabla \mathbb{P}\nabla \cdot \tau dx \\
&\quad - \int \nabla \mathbb{P}\nabla \cdot \tau \cdot \nabla(\partial_{11} u + \mathbb{P}(\nabla \cdot \tau) + \mathbb{P}(-u \cdot \nabla u))\, dx.
\end{aligned}$$

Inserting the last two equations in (2.7), we find

$$\|\nabla u\|_{H^1}^2 = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8, \tag{2.8}$$

where

$$
\begin{aligned}
J_1 &= -2\frac{d}{dt}\int u\cdot \mathbb{P}\nabla\cdot\tau dx - 2\frac{d}{dt}\int \nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx,\\
J_2 &= 2\int u\cdot\partial_{22}\mathbb{P}\nabla\cdot\tau\,dx + 2\int\nabla u\cdot\nabla\partial_{22}\mathbb{P}\nabla\cdot\tau\,dx,\\
J_3 &= -2\int u\cdot\mathbb{P}\nabla\cdot\tau\,dx - 2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau\,dx,\\
J_4 &= -2\int u\cdot\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\,dx - 2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\,dx,\\
J_5 &= -2\int u\cdot\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\,dx - 2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot Q(\tau,\nabla u)\,dx,\\
J_6 &= 2\int\mathbb{P}\nabla\cdot\tau\cdot\partial_{11}u\,dx + 2\int\nabla\mathbb{P}\nabla\cdot\tau\cdot\nabla\partial_{11}u\,dx,\\
J_7 &= 2\int\mathbb{P}\nabla\cdot\tau\cdot\mathbb{P}(\nabla\cdot\tau)\,dx + 2\int\nabla\mathbb{P}\nabla\cdot\tau\cdot\nabla\mathbb{P}(\nabla\cdot\tau)\,dx,\\
J_8 &= -2\int\mathbb{P}\nabla\cdot\tau\cdot\mathbb{P}(u\cdot\nabla u)\,dx - 2\int\nabla\mathbb{P}\nabla\cdot\tau\cdot\nabla\mathbb{P}(u\cdot\nabla u)\,dx.
\end{aligned}
$$

We first have

$$
\begin{aligned}
\int_0^t J_1\,ds &\le C\,\|u(t)\|_{L^2}\|\tau(t)\|_{H^1} + C\,\|u_0\|_{L^2}\|\tau_0\|_{H^1}\\
&\quad + C\,\|u(t)\|_{H^1}\|\tau(t)\|_{H^2} + C\,\|u_0\|_{H^1}\|\tau_0\|_{H^2}\\
&\le C\,\|u(t)\|_{H^1}\|\tau(t)\|_{H^2} + C\,\|u_0\|_{H^1}\|\tau_0\|_{H^2}.
\end{aligned}
$$

By integration by parts and Hölder's inequality,

$$
\begin{aligned}
|J_2| &\le \|\nabla u\|_{H^1}\,\|\partial_2\tau\|_{H^2}, \qquad |J_3| \le \|\nabla u\|_{H^1}\|\tau\|_{H^1},\\
|J_6| &\le \|\nabla\tau\|_{H^1}\,\|\partial_1 u\|_{H^2}, \qquad |J_7| \le \|\nabla\cdot\tau\|_{H^1}^2 \le \|\tau\|_{H^2}^2.
\end{aligned}
$$

By integration by parts, Hölder's inequality and Lemma 2.1,

$$
\begin{aligned}
|J_4| &\le \|\nabla u\|_{L^2}\,\|u\|_{L^\infty}\|\nabla\tau\|_{L^2} + C\,\|\Delta u\|_{L^2}\,\|\nabla u\|_{L^2}^{\frac12}\,\|\partial_1\nabla u\|_{L^2}^{\frac12}\,\|\nabla\tau\|_{L^2}^{\frac12}\,\|\partial_2\nabla\tau\|_{L^2}^{\frac12}\\
&\quad + C\,\|\Delta u\|_{L^2}\,\|u\|_{L^\infty}\|\Delta\tau\|_{L^2}\\
&\le C\,\|u\|_{H^2}(\|\nabla u\|_{H^1}^2 + \|\nabla\tau\|_{H^1}^2) + C\,\|\nabla u\|_{L^2}^{\frac12}\,\|\nabla\tau\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}\|\partial_1 u\|_{H^2}^{\frac12}\,\|\partial_2\tau\|_{H^2}^{\frac12}\\
&\le C\,(\|u\|_{H^2} + \|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2 + \|\partial_1 u\|_{H^2}^2 + \|\nabla\tau\|_{H^1}^2 + \|\partial_2\tau\|_{H^2}^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|J_5| &\le \|\nabla u\|_{L^2}\,\|\tau\|_{L^\infty}\|\nabla u\|_{L^2} + C\,\|\Delta u\|_{L^2}\,\|\nabla u\|_{L^2}^{\frac12}\,\|\partial_1\nabla u\|_{L^2}^{\frac12}\,\|\nabla\tau\|_{L^2}^{\frac12}\,\|\partial_2\nabla\tau\|_{L^2}^{\frac12}\\
&\quad + C\,\|\Delta u\|_{L^2}\,\|\tau\|_{L^\infty}\|\Delta u\|_{L^2}\\
&\le C\,\|\tau\|_{H^2}\,\|\nabla u\|_{H^1}^2 + C\,\|\nabla u\|_{L^2}^{\frac12}\,\|\nabla\tau\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}\|\partial_1 u\|_{H^2}^{\frac12}\,\|\partial_2\tau\|_{H^2}^{\frac12}\\
&\le C\,(\|u\|_{H^2} + \|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2 + \|\partial_1 u\|_{H^2}^2 + \|\nabla\tau\|_{H^1}^2 + \|\partial_2\tau\|_{H^2}^2)
\end{aligned}
$$

and

$$\begin{aligned}|J_8| &\le 2\|\nabla\tau\|_{L^2}\|u\|_{L^\infty}\,\|\nabla u\|_{L^2} + C\|\Delta\tau\|_{L^2}^{\frac12}\,\|\partial_2\Delta\tau\|_{L^2}^{\frac12}\,\|\nabla u\|_{L^2}^{\frac32}\|\partial_1\nabla u\|_{L^2}^{\frac12}\\
&\quad + C\,\|\Delta\tau\|_{L^2}^{\frac12}\,\|\partial_2\Delta\tau\|_{L^2}^{\frac12}\,\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}\\
&\le C\,\|u\|_{H^2}\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^2}\\
&\quad + C\,(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2\tau\|_{H^2}^2)\\
&\le C\,(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}+\|\partial_2\tau\|_{H^2}^2).\end{aligned}$$

Inserting the bounds above in (2.8) and integrating in time, we obtain

$$\begin{aligned}E_2(t) := \int_0^t \|\nabla u(s)\|_{L^2}^2\,ds &= -2\int u\cdot\mathbb{P}\nabla\cdot\tau dx + 2\int u_0\cdot\mathbb{P}\nabla\cdot\tau_0\,dx\\
&\quad -2\int \nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx + 2\int \nabla u_0\cdot\nabla\mathbb{P}\nabla\cdot\tau_0\,dx\\
&\quad + \int_0^t (J_1+J_2+\cdots+J_8)\,ds\\
&\le C\,\|u(t)\|_{H^1}\|\tau(t)\|_{H^2} + C\,\|u_0\|_{H^1}\|\tau_0\|_{H^2}\\
&\quad + CE_1(t) + \frac12 E_2(t) + CE_1^{\frac32}(t) + CE_2^{\frac32}(t),\\
&\le CE(0) + CE_1(t) + \frac12 E_2(t) + CE_1^{\frac32}(t) + CE_2^{\frac32}(t),\end{aligned} \tag{2.9}$$

where we have used several Hölder's inequalities,

$$\begin{aligned}\int_0^t \|\nabla u\|_{H^1}\,\|\partial_2\tau\|_{H^2}ds &\le \frac14\int_0^t \|\nabla u\|_{H^1}^2\,ds + C\int_0^t \|\partial_2\tau\|_{H^2}^2 ds\\
&\le \frac14 E_2(t) + C\,E_1(t),\end{aligned}$$

$$\int_0^t \|\nabla u\|_{H^1}\|\tau\|_{H^1}\,ds \le \frac14 E_2(t) + C\,E_1(t),$$

$$\int_0^t \|\nabla\tau\|_{H^1}\,\|\partial_1 u\|_{H^2} \le C\;E_1(t), \qquad \int_0^t \|\nabla\cdot\tau\|_{H^1}^2 ds \le \|\tau\|_{H^2}^2\,ds \le C\;E_1(t)$$

and

$$\begin{aligned}&\int_0^t (\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2+\|\partial_1 u\|_{H^2}^2+\|\tau\|_{H^2}+\|\partial_2\tau\|_{H^2}^2)\,ds\\
&\le CE_1^{\frac32}(t) + CE_2^{\frac32}(t).\end{aligned}$$

It then follows from (2.9) that

$$\frac12 E_2(t) \le CE(0) + CE_1(t) + CE_1^{\frac32}(t) + CE_2^{\frac32}(t),$$

which is (2.2). This completes the proof of Theorem 1.1. □

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