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Stability of 3D perturbations near a special 2D solution to the rotating Boussinesq equations

Liangliang Ma[1] | **Jiahong Wu**[2] | **Qian Zhang**[3]

[1] College of Mathematics and Physics, Chengdu University of Technology, Chengdu, China

[2] Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma, USA

[3] Hebei Key Laboratory of Machine Learning and Computational Intelligence, School of Mathematics and Information Science, Hebei University, Baoding, China

Correspondence
Qian Zhang, Hebei Key Laboratory of Machine Learning and Computational Intelligence, School of Mathematics and Information Science, Hebei University, Baoding, 071002, China.
Email: zhangqian@hbu.edu.cn

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Abstract

The three-dimensional (3D) rotating Boussinesq system governs the dynamics of fluid flows in the atmosphere and the ocean. It admits a special two-dimensional (2D) solution which involves only vertical motion and density changes that both vary only in the horizontal plane. This solution illustrates the important effect of gravity and leads to the renowned Brunt–Väisälä frequency. A natural question with practical applications is whether or not general 3D perturbations near this 2D solution are stable. This paper establishes the stability of any 3D perturbations even when the Boussinesq system involves only horizontal dissipation and horizontal thermal conduction. An anisotropic Sobolev setting is selected to reduce the regularity assumption on the initial data.

KEYWORDS
hydrostatic balance, rotating Boussinesq equations, stability

1 | INTRODUCTION

The dynamics of fluid flows in the atmosphere and the ocean is controlled by the interaction of gravity and the earth's rotation with the density variations about a reference state. When the fluid velocities are too slow to involve compressible effects, the model at work is the incompressible rotating Boussinesq system.[1,2] Mathematically, the Boussinesq equations can be written as

$$\begin{cases} \partial_t v + v\cdot\nabla v + f\, e_3 \times v = -\nabla p^* + \nu\Delta v - \frac{g}{\rho_b}\rho^* e_3, \quad x\in\mathbb{R}^3, t>0,\\ \nabla\cdot v = 0,\\ \partial_t \rho^* + v\cdot\nabla\rho^* = \kappa\Delta\rho^*, \end{cases} \tag{1}$$

where $v = (v_1, v_2, v_3)$ denotes the fluid velocity, p^* pressure, $e_3 = (0,0,1)$, ρ^* is the density, ρ_b is the reference constant density, $\nu \geq 0$ is the coefficient of viscosity, $\kappa \geq 0$ is the coefficient of heat conduction, and $f = f(x_2)$ denotes the rotation frequency. We will assume that f is smooth with compact support.

For many scales of fluids in the atmosphere and in the ocean, the pressure gradients nearly cancel the buoyancy term on the right of (1), namely,

$$-\nabla p^* - \frac{g}{\rho_b}\rho^* e_3 = 0 \tag{2}$$

or

$$-\frac{\partial p^*}{\partial x_3} = \frac{g}{\rho_b}\rho^*. \tag{3}$$

Together with $v = 0$, (3) is an exact solution of (1) when the pressure $p^*(x_3)$ is a quadratic function of x_3 in the case of full dissipation. This special solution is referred to as hydrostatic balance. It is natural to consider perturbations about a mean state in hydrostatic balance $(\bar{p}, \bar{\rho})$, where $\bar{p}$ and $\bar{\rho}$ are in hydrostatic balance

$$\frac{\partial}{\partial x_3}\bar{p}(x_3) = -\frac{g\bar{\rho}}{\rho_b}. \tag{4}$$

Then, the perturbation (v, p, ρ) with

$$p = p^* - \bar{p}(x_3), \quad \rho(x,t) = \rho^* - \bar{\rho}(x_3) \tag{5}$$

solves

$$\begin{cases} \partial_t v + v\cdot\nabla v + f\, e_3\times v = -\nabla p + \nu\Delta v - \frac{g\rho}{\rho_b}e_3,\\ \nabla\cdot v = 0,\\ \partial_t\rho + v\cdot\nabla\rho + \frac{d\bar{\rho}}{dx_3}v_3 = \kappa\Delta\rho + \kappa\partial_{x_3x_3}\bar{\rho}(x_3). \end{cases} \tag{6}$$

Equation (6) is the fundamental model for fluids in the atmosphere and in the ocean. When $\frac{d\bar{\rho}}{dx_3} < 0$, the situation is stable. Higher up in the atmosphere and in the deep ocean, the lighter fluid sits atop heavier fluid in a stable situation.

We assume

$$g = \rho_b = 1 \quad \text{and} \quad \bar{\rho}(x_3) = -x_3. \tag{7}$$

In addition, we assume there is only horizontal dissipation, even though all the results remain valid for the full dissipation case. That is, our focus will be on the following Boussinesq system:

$$\begin{cases} \partial_t v + v \cdot \nabla v + f\, e_3 \times v = -\nabla p + \nu \Delta_h v - \rho e_3, \\ \nabla \cdot v = 0, \\ \partial_t \rho + v \cdot \nabla \rho - v_3 = \kappa \Delta_h \rho, \end{cases} \tag{8}$$

where $\Delta_h = \partial_1^2 + \partial_2^2$. Equation (8) admits an exact special solution, which represents the viscous version of the solution expressed in terms of the Brunt–Väisälä frequency. To make the process of verifying the special solution more transparent, we distinguish the difference between the horizontal and the vertical components. We set

$$v = (v_h, w), \quad v_h = (v_1, v_2), \quad v_h^{\perp} = (-v_2, v_1), \tag{9}$$

$$\nabla_h = (\partial_1, \partial_2), \quad \nabla_h^{\perp} = (-\partial_2, \partial_1). \tag{10}$$

Equation (8) can then be rewritten as

$$\begin{cases} \partial_t v_h + v_h \cdot \nabla_h v_h + w \partial_3 v_h + f v_h^{\perp} = -\nabla_h p + \nu \Delta_h v_h, \\ \partial_t w + v_h \cdot \nabla_h w + w \partial_3 w = -\partial_3 p - \rho + \nu \Delta_h w, \\ \nabla_h \cdot v_h + \partial_3 w = 0, \\ \partial_t \rho + v_h \cdot \nabla_h \rho + w \partial_3 \rho - w = \kappa \Delta_h \rho. \end{cases} \tag{11}$$

A special solution of (11) is given by

$$(v_h^{(0)}, w^{(0)}, \rho^{(0)}) = (0, w^{(0)}(x_h, t), \rho^{(0)}(x_h, t)), \quad p^{(0)} = 0 \tag{12}$$

with $(w^{(0)}, \rho^{(0)})$ satisfying

$$\begin{cases} \partial_t w^{(0)} = \nu \Delta_h w^{(0)} - \rho^{(0)}, \\ \partial_t \rho^{(0)} = \kappa \Delta_h \rho^{(0)} + w^{(0)}, \\ w^{(0)}(x_h, 0) = w_0^{(0)}(x_h), \quad \rho^{(0)}(x_h, 0) = \rho_0^{(0)}(x_h). \end{cases} \tag{13}$$

This exact solution is two-dimensional (2D). This special 2D solution illustrates the effect of gravity and, in the case of no dissipation or heat conduction, yields the solution in terms of the renowned Brunt–Väisälä frequency, which measures the atmospheric stratification.

This paper intends to understand the stability properties of general three-dimensional (3D) perturbations near this exact 2D solution. The study of this stability problem is very important and fundamental to environmental fluid dynamics (see, e.g., Refs. 3, 4). This paper establishes some of observed weather phenomena as rigorous mathematical facts. We assume either full dissipation and heat conduction or horizontal dissipation and heat conduction. Mathematically, the horizontal dissipation case is more challenging, and the results and techniques for the horizontal dissipation case remain valid for the full dissipation case. Therefore, to reduce the redundancy, we shall only present the details for the horizontal dissipation case.

When the dissipation is anisotropic and only in the horizontal direction, the Boussinesq system is of mixed type, parabolic in the horizontal direction but hyperbolic in the vertical direction. In order to reduce the regularity requirement on the initial data, the most suitable functional setting appears to be an anisotropic Sobolev space. Let $\mathcal{S}'$ be the space of tempered distributions. Let σ_1 and σ_2 be two real numbers. The inhomogeneous anisotropic Sobolev space is defined as follows:

$$H^{\sigma_1,\sigma_2} = \left\{ f \in \mathcal{S}' \,:\, \|f\|_{H^{\sigma_1,\sigma_2}} < \infty \right\}, \tag{14}$$

where

$$\|f\|_{H^{\sigma_1,\sigma_2}} := \left[\int_{\mathbb{R}^3} \left(1 + |\xi_h|^2\right)^{\sigma_1} \left(1 + |\xi_3|^2\right)^{\sigma_2} |\widehat{f}(\xi)|^2 \, d\xi \right]^{\frac{1}{2}}. \tag{15}$$

Here, $\widehat{f}$ is the Fourier transform of f. Anisotropic Sobolev spaces are appropriate functional settings for anisotropic differential equations such as the anisotropic Navier–Stokes equations.[5] It is worth remarking that the anisotropic Sobolev space H^{σ_1,σ_2} defined here corresponds to $H^{\sigma_1}(\mathbb{R}^2, H^{\sigma_2}(\mathbb{R}))$, and H^{σ_1,σ_2} with $\sigma_1 = \sigma_2$ is not the same as $H^{\sigma_1}(\mathbb{R}^3)$. We focus on (11) supplemented with the initial data

$$v_h(x,0) = v_{0h}(x), \quad w(x,0) = w_0(x), \quad \rho(x,0) = \rho_0(x). \tag{16}$$

Our main conclusion is the following global existence, uniqueness and stability result.

Theorem 1. *Consider (11) with $\nu > 0$ and $\kappa > 0$. Let $(v_{0h}, w_0, \rho_0) \in H^{0,1}(\mathbb{R}^3)$ with $\nabla_h \cdot v_{0h} + \partial_3 w_0 = 0$. Assume the initial data $(w_0^{(0)}, \rho_0^{(0)})$ for the special 2D solution in (13) is in $L^2(\mathbb{R}^2)$. Then, there exists a constant $\varepsilon_0 = \varepsilon_0(\nu, \kappa) > 0$ such that, if*

$$\|(v_{0h}, w_0, \rho_0) - (0, w_0^{(0)}, \rho_0^{(0)})\|_{H^{0,1}} \le \varepsilon \tag{17}$$

for some $\varepsilon \le \varepsilon_0$, then (11) has a unique global solution

$$(v_h, w, \rho) \in C([0,\infty); L^2), \quad (v_h, w, \rho) \in L^\infty(0,\infty; H^{0,1}),$$
$$(\nabla_h v_h, \nabla_h w, \nabla_h \rho) \in L^2(0,\infty; H^{0,1}). \tag{18}$$

In addition, for a constant $C > 0$ (independent of ν and κ) and for any $t > 0$,

$$\|(v_h, w, \rho)(t) - (0, w^{(0)}, \rho^{(0)})(t)\|_{H^{0,1}} \leq C\,\varepsilon, \tag{19}$$

where $(w^{(0)}, \rho^{(0)})$ is the special 2D solution of (13).

Theorem 1 states that, when (v_{0h}, w_0, ρ_0) is close to $(0, w_0^{(0)}, \rho_0^{(0)})$, then (11) has a unique global solution that remains close to the special 2D solution given by (13) for all time. In the modeling of air parcels in the atmosphere, the 2D structure of the special solution may not be exact 2D and may contain 3D structures as perturbations, and this theorem assures us that the perturbed 2D structure is stable. In addition, this stability result is different from many of the stability results for which the special solution is steady and independent of time. The special solution involved here is dynamic and requires additional treatment.

The proof of Theorem 1 consists of two main parts. The first part establishes a global a priori bound of the solution in $H^{0,1}$ while the second obtains the local existence and uniqueness. The global bound is shown via a bootstrapping argument (see, e.g., Ref. 6). The process starts with the equations of the difference $(v_h, \widetilde{w}, \widetilde{\rho})$, where

$$\widetilde{w} = w - w^{(0)}, \quad \widetilde{\rho} = \rho - \rho^{(0)}. \tag{20}$$

It is easy to check that $(v_h, \widetilde{w}, \widetilde{\rho})$ satisfies

$$\begin{cases} \partial_t v_h + v_h \cdot \nabla_h v_h + \widetilde{w}\,\partial_3 v_h + w^{(0)}\partial_3 v_h + f v_h^{\perp} = -\nabla_h p + \nu \Delta_h v_h, \\ \partial_t \widetilde{w} + v_h \cdot \nabla_h \widetilde{w} + \widetilde{w}\partial_3 \widetilde{w} + v_h \cdot \nabla_h w^{(0)} + w^{(0)}\partial_3 \widetilde{w} = -\partial_3 p - \widetilde{\rho} + \nu \Delta_h \widetilde{w}, \\ \nabla_h \cdot v_h + \partial_3 \widetilde{w} = 0, \\ \partial_t \widetilde{\rho} + v_h \cdot \nabla_h \widetilde{\rho} + \widetilde{w}\partial_3 \widetilde{\rho} + v_h \cdot \nabla_h \rho^{(0)} + w^{(0)}\partial_3 \widetilde{\rho} - \widetilde{w} = \kappa \Delta_h \widetilde{\rho}, \\ (v_h, \widetilde{w}, \widetilde{\rho})|_{t=0} = (v_{0h}, \widetilde{w}_0, \widetilde{\rho}_0) := (v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)}). \end{cases} \tag{21}$$

For a solution $(v_h, \widetilde{w}, \widetilde{\rho})$ of (21), we define the energy functional

$$\begin{aligned} E(t) = &\sup_{0\leq\tau\leq t} \|(v_h, \widetilde{w}, \widetilde{\rho})(\tau)\|_{H^{0,1}}^2 \\ &+ \nu \int_0^t \|(v_h, \widetilde{w})(\tau)\|_{H^{1,1}}^2\, d\tau + \kappa \int_0^t \|\widetilde{\rho}(\tau)\|_{H^{1,1}}^2\, d\tau. \end{aligned} \tag{22}$$

Our main efforts are then devoted to proving the following inequality, for any $t > 0$:

$$E(t) \leq K_0 E(0) + C\,(\nu^{-4} + \nu^{-1}\kappa^{-3})\,K_0\,E(t)^3, \tag{23}$$

where C is an absolute constant independent of ν and κ, and

$$K_0 := e^{C\,(\nu^{-2} + \nu^{-\frac{1}{2}}\kappa^{-\frac{3}{2}})\,\|(w_0^{(0)}\rho_0^{(0)})\|_{L^2(\mathbb{R}^2)}^2}. \tag{24}$$

A direct application of the bootstrapping argument concludes that, if

$$E(0) := \|(v_{0h}, \widetilde{w}_0, \widetilde{\rho}_0)\|^2_{H^{0,1}} \le \varepsilon \tag{25}$$

for sufficiently small $\varepsilon > 0$, then, for a constant $C > 0$ and for all $0 < t < \infty$,

$$E(t) \le C\,\varepsilon, \tag{26}$$

which, especially, yields the desired global bound on the solution $\|(v_h, \widetilde{w}, \widetilde{\rho})(\tau)\|_{H^{0,1}}$. We briefly comment on how ε depends on ν and κ. The energy inequality in (23) and the bootstrapping argument hint at how ε depends on ν and κ. The explicit coefficients in (23) allow us to deduce that ε depends on ν^2, $\nu^{\frac{1}{2}}\kappa^{\frac{3}{2}}$, and K_0. We note that K_0 also depends on ν and κ. The dependence of ε on ν^2 and $\nu^{\frac{1}{2}}\kappa^{\frac{3}{2}}$ may be optimal, but it is extremely laborious to verify this since so many terms are involved in the estimates.

The proof of (23) is not trivial. We fully exploit the structure of the system and make use of several anisotropic Sobolev inequalities to take advantage of the anisotropic dissipation. Compared with the isotropic dissipation case, it is much more delicate to bound the nonlinear terms when we only have anisotropic dissipation. In view of the anisotropic dissipation, controlling the nonlinearity is highly nontrivial and requires different treatments on horizontal and vertical derivatives. Anisotropic Sobolev inequalities instead of the standard ones are employed here to generate favorable derivatives. More technical details are provided in Section 2.

The local existence and uniqueness part is shown via Friedrichs' method. Its implementation consists of three steps. The first step constructs a sequence of approximate solutions $\{(v_h^{(n)}, \widetilde{w}^{(n)}, \widetilde{\rho}^{(n)})\}_{n\in\mathbb{N}}$ to regularized systems. These regularized systems result from the Fourier cutoff of the terms in (21). For each fixed $n \in \mathbb{N}$, the global (in time) existence and uniqueness is a consequence of Bernstein's inequality, and the existence and extension theory for ordinary differential equations on Banach spaces. The second step is to establish uniform (in n) local bounds on $(v_h^{(n)}, \widetilde{w}^{(n)}, \widetilde{\rho}^{(n)})$ in the functional setting

$$L^\infty(0,T;H^{0,1}) \cap L^2(0,T;H^{1,1}) \tag{27}$$

for a uniform time interval $[0,T]$. This step is very involved and is accomplished by decomposing $(v_h^{(n)}, \widetilde{w}^{(n)}, \widetilde{\rho}^{(n)})$ into the sum of free solutions and the remainder. The third step is to show that a subset of $(v_h^{(n)}, \widetilde{w}^{(n)}, \widetilde{\rho}^{(n)})$ converges to $(v_h, \widetilde{w}, \widetilde{\rho})$, a solution of (21). This step invokes the Aubin–Lions lemma. More technical details are left to Section 3. Due to the weak functional setting of the solutions, the uniqueness is not obvious and its proof requires repeated applications of the anisotropic inequalities.

We mention some related work. Due to their meteorological applications and mathematical significance, stability and large-time behavior problems on the Boussinesq equations near several important special solutions have attracted considerable interests. In particular, two classes of steady states, hydrostatic balance and shear flows, are among the most prominent special solutions to the Boussinesq equations. Mathematically, the systems governing the perturbations near these steady states exhibit special properties such as enhanced dissipation and stabilizing phenomenon. These properties have recently been discovered and exploited to solve several seemingly impossible stability problems (see Refs. 7–23). One main difference between our stability result presented here and the previous work is that the 2D special solution focused here is time-dependent. This special solution represents the viscous version of the Brunt–Väisälä solution, which models the stable vertical oscillation near the original location on average. This is the situation typically satisfied on both the atmosphere and ocean once one moves a sufficient distance from the atmosphere/ocean/land boundary.[1] Our main result asserts the robustness of this special 2D solution and the global stability under any 3D perturbations. We would like to mention some important investigations of Stechmann and his collaborators on several aspects of the Boussinesq systems near the hydrostatic balance such as the interaction between slow and fast modes, and the energy decompositions (see Refs. 24, 25). In addition, there have been significant developments on many well-posedness and related problems (see, e.g., Refs. 6, 7, 16–19, 26–55). This list is by no means exhaustive.

The rest of this paper is divided into two main sections and an Appendix. Section 2 establishes the global (in time) bound on the solution while Section 3 prove the local existence and uniqueness of the solution. The Appendix provides several facts that have been used in the first two sections.

2 | GLOBAL A PRIORI BOUND

This section proves the energy inequality (23). More precisely, we establish the following proposition.

Proposition 1. *Assume that $(v_h, \widetilde{w}, \widetilde{\rho})$ solves (21). Let $E(t)$ be defined as in (22), namely,*

$$E(t) = \sup_{0\le\tau\le t} \|(v_h, \widetilde{w}, \widetilde{\rho})(\tau)\|_{H^{0,1}}^2 + \nu \int_0^t \|(v_h, \widetilde{w})(\tau)\|_{H^{1,1}}^2\, d\tau + \kappa \int_0^t \|\widetilde{\rho}(\tau)\|_{H^{1,1}}^2\, d\tau. \tag{28}$$

Then $E(t)$ satisfies, for any $0 < t < \infty$,

$$E(t) \le K_0 E(0) + C\,(\nu^{-4} + \nu^{-2}\kappa^{-2} + \nu^{-1}\,\kappa^{-3})\,K_0\,E(t)^3, \tag{29}$$

where C is an absolute constant independent of ν and κ, and

$$K_0 := e^{C\,(\nu^{-2}+\nu^{-\frac{1}{2}}\,\kappa^{-\frac{3}{2}})\,\|(w_0^{(0)}\rho_0^{(0)})\|_{L^2(\mathbb{R}^2)}^2}. \tag{30}$$

To prove Proposition 1, we need the anisotropic upper bounds for triple products. Such inequalities and their proofs can be found in several references.[5,28 36,56]

Lemma 1. *There exists a constant $C > 0$ such that*

$$\left|\int_{\mathbb{R}^3} F(x)\,G(x)\,H(x)\,dx\right| \le C\,\|F\|_{L^2}^{\frac{1}{2}}\,\|\partial_1 F\|_{L^2}^{\frac{1}{2}}\,\|G\|_{L^2}^{\frac{1}{2}}\,\|\partial_2 G\|_{L^2}^{\frac{1}{2}}\,\|H\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 H\|_{L^2}^{\frac{1}{2}},$$

$$\left|\int_{\mathbb{R}^3} F(x)\,G(x)\,H(x)\,dx\right| \le C\,\|F\|_{L^2}^{\frac{1}{4}}\,\|\partial_3 F\|_{L^2}^{\frac{1}{4}}\,\|\nabla_h F\|_{L^2}^{\frac{1}{4}}\,\|\nabla_h\partial_3 F\|_{L^2}^{\frac{1}{4}} \times \|G\|_{L^2}\,\|H\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h H\|_{L^2}^{\frac{1}{2}}. \tag{31}$$

We are now ready to prove Proposition 1.

Proof of Proposition 1. To simplify the notation, we remove tilde and write w and ρ for $\widetilde{w}$ and $\widetilde{\rho}$, respectively. We focus on the system satisfied by (v_h, w, ρ),

$$\begin{cases}\partial_t v_h + v_h \cdot \nabla_h v_h + w^{(0)}\partial_3 v_h + w\,\partial_3 v_h + f v_h^{\perp} = -\nabla_h p + \nu\Delta_h v_h,\\ \partial_t w + v_h \cdot \nabla_h w^{(0)} + v_h \cdot \nabla_h w + w\partial_3 w + w^{(0)}\partial_3 w = -\partial_3 p - \rho + \nu\Delta_h w,\\ \nabla_h \cdot v_h + \partial_3 w = 0,\\ \partial_t \rho + v_h \cdot \nabla_h \rho + v_h \cdot \nabla_h \rho^{(0)} + w\partial_3 \rho + w^{(0)}\partial_3 \rho - w = \kappa\Delta_h \rho,\\ (v_h, w, \rho)|_{t=0} = (v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)}).\end{cases} \tag{32}$$

First, we recall that $(w^{(0)}, \rho^{(0)}) = (w^{(0)}(x_h, t), \rho^{(0)}(x_h, t))$ satisfies the following 2D system:

$$\begin{cases}\partial_t w^{(0)} = \nu\Delta_h w^{(0)} - \rho^{(0)},\\ \partial_t \rho^{(0)} = \kappa\Delta_h \rho^{(0)} + w^{(0)},\\ w^{(0)}(x_h, 0) = w_0^{(0)}(x_h), \quad \rho^{(0)}(x_h, 0) = \rho_0^{(0)}(x_h)\end{cases} \tag{33}$$

with $(w_0^{(0)}, \rho_0^{(0)}) \in L^2(\mathbb{R}^2)$. Dotting (33) with $(w^{(0)}, \rho^{(0)})$ and integrating in time yields

$$\begin{aligned}&\|(w^{(0)}, \rho^{(0)})(t)\|_{L^2(\mathbb{R}^2)}^2 + 2\nu\int_0^t \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2\,d\tau + 2\kappa\int_0^t \|\nabla_h \rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2\,d\tau\\ &= \|(w_0^{(0)} \rho_0^{(0)})\|_{L^2(\mathbb{R}^2)}^2.\end{aligned} \tag{34}$$

Testing the equations of (v_h, w, ρ) in (32) with (v_h, w, ρ) yields

$$\begin{aligned}&\frac{d}{dt}(\|v_h\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\rho\|_{L^2}^2) + 2\nu\|\nabla_h v_h\|_{L^2}^2 + 2\nu\|\nabla_h w\|_{L^2}^2 + 2\kappa\|\nabla_h \rho\|_{L^2}^2\\ &= -2\int v_h \cdot \nabla_h w^{(0)} w\,dx - 2\int v_h \cdot \nabla_h \rho^{(0)}\,\rho\,dx,\end{aligned} \tag{35}$$

where we have used the divergence-free condition and a simple fact

$$\nabla_h \cdot v_h + \partial_3 w = 0, \qquad v_h^{\perp} \cdot v_h = 0. \tag{36}$$

In addition, we have used the fact that $w^{(0)}(x_h, t)$ only depends on the horizontal variable to obtain

$$\int w^{(0)}\partial_3 v_h \cdot v_h\,dx = 0, \quad \int w^{(0)}\partial_3 w\,w\,dx = 0, \quad \int w^{(0)}\partial_3 \rho\,\rho\,dx = 0. \tag{37}$$

It is worth remarking that the Coriolis forcing term does not contribute to the L^2-norm. The two terms in (35) can be bounded as follows. Noticing that $w^{(0)}$ and $\rho^{(0)}$ depend only on x_h, applying

Hölder's inequality and Ladyzhenskaya's inequality, we have

$$
\begin{aligned}
\left|\int v_h \cdot \nabla_h w^{(0)} w\, dx\right| \leq& \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)} \|v_h\|_{L^2_{x_3}L^4_h} \|w\|_{L^2_{x_3}L^4_h} \\
\leq& C\,\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)} \|v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla_h w\|_{L^2}^{\frac{1}{2}} \\
\leq& \frac{\nu}{4}\|\nabla_h v_h\|_{L^2}^2 + \frac{\nu}{4}\|\nabla_h w\|_{L^2}^2 \\
&+ C\,\nu^{-1}\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 (\|v_h\|_{L^2}^2 + \|w\|_{L^2}^2). \qquad (38)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left|\int v_h \cdot \nabla_h \rho^{(0)} \rho\, dx\right| \leq& \frac{\nu}{4}\|\nabla_h v_h\|_{L^2}^2 + \frac{\kappa}{4}\|\nabla_h \rho\|_{L^2}^2 \\
&+ C\,\nu^{-\frac{1}{2}}\,\kappa^{-\frac{1}{2}}\|\nabla_h \rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2 (\|v_h\|_{L^2}^2 + \|\rho\|_{L^2}^2). \qquad (39)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}(\|v_h\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\rho\|_{L^2}^2) + \nu\|\nabla_h v_h\|_{L^2}^2 + \nu\|\nabla_h w\|_{L^2}^2 + \kappa\|\nabla_h \rho\|_{L^2}^2 \\
&\leq C\,(\nu^{-1}\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 + \nu^{-\frac{1}{2}}\,\kappa^{-\frac{1}{2}}\|\nabla_h \rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2)(\|v_h\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\rho\|_{L^2}^2). \qquad (40)
\end{aligned}
$$

We now estimate $\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2 + \|\partial_3 \rho\|_{L^2}^2$. Applying ∂_3 to the equations of (v_h, w, ρ) in (32) and testing with $(\partial_3 v_h, \partial_3 w, \partial_3 \rho)$, we have

$$
\begin{aligned}
&\frac{d}{dt}(\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2 + \|\partial_3 \rho\|_{L^2}^2) \\
&+ 2\nu\|\nabla_h \partial_3 v_h\|_{L^2}^2 + 2\nu\|\nabla_h \partial_3 w\|_{L^2}^2 + 2\kappa\|\nabla_h \partial_3 \rho\|_{L^2}^2 \\
&= I_1 + \cdots + I_8, \qquad (41)
\end{aligned}
$$

where

$$I_1 = -\int \partial_3(v_h \cdot \nabla_h v_h + w\,\partial_3 v_h) \cdot \partial_3 v_h\, dx,$$

$$I_2 = -\int \partial_3(v_h \cdot \nabla_h w + w\,\partial_3 w) \cdot \partial_3 w\, dx,$$

$$I_3 = -\int \partial_3(v_h \cdot \nabla_h \rho + w\,\partial_3 \rho) \cdot \partial_3 \rho\, dx,$$

$$I_4 = -\int \partial_3(w^{(0)} \partial_3 v_h) \cdot \partial_3 v_h\, dx,$$

$$I_5 = -\int \partial_3(v_h \cdot \nabla_h w^{(0)})\, \partial_3 w\, dx,$$

$$I_6 = -\int \partial_3(w^{(0)} \cdot \partial_3 w)\,\partial_3 w\,dx,$$

$$I_7 = -\int \partial_3(v_h \cdot \nabla_h \rho^{(0)})\,\partial_3 \rho\,dx,$$

$$I_8 = -\int \partial_3(w^{(0)} \cdot \partial_3 \rho)\,\partial_3 \rho\,dx. \tag{42}$$

We have already used the following simple facts:

$$\partial_3 v_h^{\perp} \cdot \partial_3 v_h = 0,$$
$$\int (\partial_3 \nabla_h p \cdot \partial_3 v_h + \partial_3^2 p \partial_3 w)\,dx = -\int \partial_3 p\,\partial_3(\nabla_h \cdot v_h + \partial_3 w)\,dx = 0. \tag{43}$$

We now estimate the terms I_1 through I_8 and start with I_1. By the divergence-free condition $\nabla_h \cdot v_h + \partial_3 w = 0$,

$$I_1 = -\int \partial_3 v_h \cdot \nabla_h v_h \cdot \partial_3 v_h\,dx - \int \partial_3 w\,\partial_3 v_h \cdot \partial_3 v_h\,dx$$
$$= -\int \partial_3 v_h \cdot \nabla_h v_h \cdot \partial_3 v_h\,dx + \int \nabla_h \cdot v_h \partial_3 v_h \cdot \partial_3 v_h\,dx. \tag{44}$$

By the anisotropic upper bounds in Lemma 1,

$$|I_1| \le C\,\|\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_3 v_h\|_{L^2}^{\frac{1}{2}}$$
$$+ C\,\|\nabla_h \cdot v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3\nabla_h \cdot v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_3 v_h\|_{L^2}^{\frac{1}{2}}$$
$$\le C\,\|\partial_3\nabla_h v_h\|_{L^2}^{\frac{3}{2}}\,\|\nabla_h v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 v_h\|_{L^2}$$
$$\le \frac{\nu}{8}\|\partial_3\nabla_h v_h\|_{L^2}^2 + C\,\nu^{-3}\|\nabla_h v_h\|_{L^2}^2\,\|\partial_3 v_h\|_{L^2}^4, \tag{45}$$

where C is an absolute constant independent of ν. The estimate of I_2 is similar.

$$I_2 = -\int \partial_3 v_h \cdot \nabla_h w\,\partial_3 w\,dx - \int \partial_3 w\,\partial_3 w\,\partial_3 w\,dx$$
$$= -\int \partial_3 v_h \cdot \nabla_h w\,\partial_3 w\,dx + \int \nabla_h \cdot v_h\,(\partial_3 w)^2\,dx. \tag{46}$$

By Lemma 1,

$$|I_2| \le C\,\|\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\partial_3 v_h\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h w\|_{L^2}^{\frac{1}{2}}\,\|\partial_3\nabla_h w\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 w\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_3 w\|_{L^2}^{\frac{1}{2}}$$
$$+ C\,\|\nabla_h \cdot v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3\nabla_h \cdot v_h\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 w\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\partial_3 w\|_{L^2}^{\frac{1}{2}}\,\|\partial_3 w\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_3 w\|_{L^2}^{\frac{1}{2}}$$

$$\leq C\,\|\nabla_h\partial_3 v_h\|_{L^2}^{\frac12}\,\|\nabla_h\partial_3 w\|_{L^2}\,\|\partial_3 v_h\|_{L^2}^{\frac12}\,\|\nabla_h w\|_{L^2}^{\frac12}\,\|\partial_3 w\|_{L^2}^{\frac12}$$

$$+\,C\,\|\nabla_h\partial_3 v_h\|_{L^2}^{\frac12}\,\|\nabla_h\partial_3 w\|_{L^2}\,\|\nabla_h v_h\|_{L^2}^{\frac12}\,\|\partial_3 w\|_{L^2}$$

$$\leq \frac{\nu}{8}\|\partial_3\nabla_h v_h\|_{L^2}^2+\frac{\nu}{8}\|\partial_3\nabla_h w\|_{L^2}^2+C\,\nu^{-3}\|\nabla_h w\|_{L^2}^2\,\|\partial_3 v_h\|_{L^2}^2\,\|\partial_3 w\|_{L^2}^2$$

$$+\,C\,\nu^{-3}\|\nabla_h v_h\|_{L^2}^2\,\|\partial_3 w\|_{L^2}^4. \tag{47}$$

To deal with I_3, we still use the divergence-free condition $\nabla_h\cdot v_h+\partial_3 w=0$ to rewrite it as

$$I_3=-\int\partial_3 v_h\cdot\nabla_h\rho\partial_3\rho\,dx+\int\nabla_h\cdot v_h\,(\partial_3\rho)^2\,dx. \tag{48}$$

Applying Lemma 1 yields

$$|I_3|\leq C\,\|\partial_3 v_h\|_{L^2}^{\frac12}\,\|\partial_1\partial_3 v_h\|_{L^2}^{\frac12}\,\|\nabla_h\rho\|_{L^2}^{\frac12}\,\|\partial_3\nabla_h\rho\|_{L^2}^{\frac12}\,\|\partial_3\rho\|_{L^2}^{\frac12}\,\|\partial_2\partial_3\rho\|_{L^2}^{\frac12}$$

$$+\,C\,\|\nabla_h\cdot v_h\|_{L^2}^{\frac12}\,\|\partial_3\nabla_h\cdot v_h\|_{L^2}^{\frac12}\,\|\partial_3\rho\|_{L^2}^{\frac12}\,\|\partial_1\partial_3\rho\|_{L^2}^{\frac12}\,\|\partial_3\rho\|_{L^2}^{\frac12}\,\|\partial_2\partial_3\rho\|_{L^2}^{\frac12}$$

$$\leq\frac{\nu}{8}\|\partial_3\nabla_h v_h\|_{L^2}^2+\frac{\kappa}{8}\|\partial_3\nabla_h\rho\|_{L^2}^2+C\,\nu^{-1}\,\kappa^{-2}\|\nabla_h\rho\|_{L^2}^2\,\|\partial_3 v_h\|_{L^2}^2\,\|\partial_3\rho\|_{L^2}^2$$

$$+\,C\,\nu^{-1}\,\kappa^{-2}\|\nabla_h v_h\|_{L^2}^2\,\|\partial_3\rho\|_{L^2}^4. \tag{49}$$

Due to the fact that $w^{(0)}=w^{(0)}(x_h,t)$ is independent of x_3,

$$I_4=-\frac12\int w^{(0)}\partial_3(|\partial_3 v_h|^2)\,dx=-\frac12\int\partial_3(w^{(0)}|\partial_3 v_h|^2)\,dx=0. \tag{50}$$

We now estimate I_5. Noticing that $\nabla_h w^{(0)}$ is independent of x_3, and applying Hölder's inequality and Ladyzhenskaya's inequality, we have

$$|I_5|=\left|\int\partial_3 v_h\cdot\nabla_h w^{(0)}\,\partial_3 w\,dx\right|$$

$$\leq\|\partial_3 v_h\|_{L^2_{x_3}L^4_h}\,\|\nabla_h w^{(0)}\|_{L^2_h}\,\|\partial_3 w\|_{L^2_{x_3}L^4_h}$$

$$\leq C\,\|\partial_3 v_h\|_{L^2}^{\frac12}\,\|\nabla_h\partial_3 v_h\|_{L^2}^{\frac12}\,\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}\,\|\partial_3 w\|_{L^2}^{\frac12}\,\|\nabla_h\partial_3 w\|_{L^2}^{\frac12}$$

$$\leq\frac{\nu}{8}\|\partial_3\nabla_h v_h\|_{L^2}^2+\frac{\nu}{8}\|\partial_3\nabla_h w\|_{L^2}^2$$

$$+\,C\,\nu^{-1}\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2\,\|\partial_3 v_h\|_{L^2}\,\|\partial_3 w\|_{L^2}. \tag{51}$$

As in I_4, we have

$$I_6=0. \tag{52}$$

The estimate of I_7 can be similarly bounded as I_5. In fact,

$$|I_7| \le \frac{\nu}{8}\|\partial_3\nabla_h v_h\|_{L^2}^2 + \frac{\kappa}{8}\|\partial_3\nabla_h\rho\|_{L^2}^2 + C\,\nu^{-\frac{1}{2}}\,\kappa^{-\frac{1}{2}}\|\nabla_h\rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2\,\|\partial_3 v_h\|_{L^2}\,\|\partial_3\rho\|_{L^2}. \quad (53)$$

As in I_4, $I_8 = 0$. We have finished the estimates of all the terms in (41). Inserting (45), (47), (49), (51), and (53) in (41) leads to

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2 + \|\partial_3\rho\|_{L^2}^2)\\
&+\nu\|\nabla_h\partial_3 v_h\|_{L^2}^2 + \nu\|\nabla_h\partial_3 w\|_{L^2}^2 + \kappa\|\nabla_h\partial_3\rho\|_{L^2}^2\\
&\le C\nu^{-3}\Big(\|\nabla_h v_h\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2\Big)\Big(\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2\Big)^2\\
&\quad + C\,\nu^{-1}\,\kappa^{-2}\Big(\|\nabla_h v_h\|_{L^2}^2 + \|\nabla_h\rho\|_{L^2}^2\Big)\Big(\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3\rho\|_{L^2}^2\Big)^2\\
&\quad + C\,\nu^{-1}\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2\,\|\partial_3 v_h\|_{L^2}\,\|\partial_3 w\|_{L^2}\\
&\quad + C\,\nu^{-\frac{1}{2}}\,\kappa^{-\frac{1}{2}}\|\nabla_h\rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2\,\|\partial_3 v_h\|_{L^2}\,\|\partial_3\rho\|_{L^2}.
\end{aligned} \quad (54)$$

Adding (40) and (54) yields

$$\begin{aligned}
&\frac{d}{dt}(\|v_h\|_{H^{0,1}}^2 + \|w\|_{H^{0,1}}^2 + \|\rho\|_{H^{0,1}}^2)\\
&\quad + \nu\|\nabla_h v_h\|_{H^{0,1}}^2 + \nu\|\nabla_h w\|_{H^{0,1}}^2 + \kappa\|\nabla_h\rho\|_{H^{0,1}}^2\\
&\le C\left(\nu^{-1}\|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 + \nu^{-\frac{1}{2}}\,\kappa^{-\frac{1}{2}}\|\nabla_h\rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2\right)\\
&\qquad \times(\|v_h\|_{H^{0,1}}^2 + \|w\|_{H^{0,1}}^2 + \|\rho\|_{H^{0,1}}^2)\\
&\quad + C\nu^{-3}\Big(\|\nabla_h v_h\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2\Big)\Big(\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2\Big)^2\\
&\quad + C\,\nu^{-1}\,\kappa^{-2}\Big(\|\nabla_h v_h\|_{L^2}^2 + \|\nabla_h\rho\|_{L^2}^2\Big)\Big(\|\partial_3 v_h\|_{L^2}^2 + \|\partial_3\rho\|_{L^2}^2\Big)^2.
\end{aligned} \quad (55)$$

The inequality in (55) is of the form

$$\frac{d}{dt}f(t) + f_1(t) \le a(t)f(t) + f_2(t). \quad (56)$$

Gronwall's inequality implies

$$f(t) + \int_0^t f_1(\tau)\,d\tau \le e^{\int_0^t a(\tau)\,d\tau} f(0) + e^{\int_0^t a(\tau)\,d\tau}\int_0^t f_2(\tau)\,d\tau. \quad (57)$$

Here

$$a(t) = C\left(\nu^{-1}\|\nabla_h w^{(0)}\|^2_{L^2(\mathbb{R}^2)} + \nu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}\|\nabla_h \rho^{(0)}\|^2_{L^2(\mathbb{R}^2)}\right) \tag{58}$$

and (34) implies that

$$e^{\int_0^t a(\tau)\,d\tau} \le e^{C(\nu^{-2}+\nu^{-\frac{1}{2}}\kappa^{-\frac{3}{2}})\|(w_0^{(0)}\rho_0^{(0)})\|^2_{L^2(\mathbb{R}^2)}} := K_0, \tag{59}$$

which depends only on ν, κ and the initial data for the special 2D solution. Here

$$\begin{aligned} f_2(t) :=& C\nu^{-3}\Big(\|\nabla_h v_h\|^2_{L^2} + \|\nabla_h w\|^2_{L^2}\Big)\Big(\|\partial_3 v_h\|^2_{L^2} + \|\partial_3 w\|^2_{L^2}\Big)^2 \\ &+ C\,\nu^{-1}\,\kappa^{-2}\Big(\|\nabla_h v_h\|^2_{L^2} + \|\nabla_h \rho\|^2_{L^2}\Big)\Big(\|\partial_3 v_h\|^2_{L^2} + \|\partial_3 \rho\|^2_{L^2}\Big)^2. \end{aligned} \tag{60}$$

It then follows from the definition of $E(t)$ in (28) that

$$\int_0^t f_2(\tau)\,d\tau \le C\,(\nu^{-4} + \nu^{-2}\kappa^{-2} + \nu^{-1}\,\kappa^{-3})\,E(t)^3. \tag{61}$$

In fact,

$$\begin{aligned} &\int_0^t f_2(\tau)\,d\tau \\ &= C\nu^{-3}\int_0^t \Big(\|\nabla_h v_h\|^2_{L^2} + \|\nabla_h w\|^2_{L^2}\Big)\Big(\|\partial_3 v_h\|^2_{L^2} + \|\partial_3 w\|^2_{L^2}\Big)^2 d\tau \\ &\quad + C\,\nu^{-1}\,\kappa^{-2}\int_0^t \Big(\|\nabla_h v_h\|^2_{L^2} + \|\nabla_h \rho\|^2_{L^2}\Big)\Big(\|\partial_3 v_h\|^2_{L^2} + \|\partial_3 \rho\|^2_{L^2}\Big)^2 d\tau \\ &\le C\nu^{-3}\sup_{0\le\tau\le t}\Big(\|\partial_3 v_h\|^2_{L^2} + \|\partial_3 w\|^2_{L^2}\Big)^2 \int_0^t \Big(\|\nabla_h v_h\|^2_{L^2} + \|\nabla_h w\|^2_{L^2}\Big)\,d\tau \\ &\quad + C\,\nu^{-1}\,\kappa^{-2}\sup_{0\le\tau\le t}\Big(\|\partial_3 v_h\|^2_{L^2} + \|\partial_3 \rho\|^2_{L^2}\Big)^2 \int_0^t \Big(\|\nabla_h v_h\|^2_{L^2} + \|\nabla_h \rho\|^2_{L^2}\Big)\,d\tau \\ &\le C\nu^{-3}E^2(t)\,\nu^{-1}E(t) + C\,\nu^{-1}\,\kappa^{-2}E^2(t)(\nu^{-1} + \kappa^{-1})E(t) \\ &= C\,(\nu^{-4} + \nu^{-2}\kappa^{-2} + \nu^{-1}\,\kappa^{-3})\,E(t)^3. \end{aligned} \tag{62}$$

Equation (57) then yields

$$E(t) \le K_0 E(0) + C\,(\nu^{-4} + \nu^{-2}\kappa^{-2} + \nu^{-1}\,\kappa^{-3})\,K_0\,E(t)^3, \tag{63}$$

where C is an absolute constant independent of ν and κ. This completes the proof of Proposition 1. ■

3 | LOCAL EXISTENCE AND UNIQUENESS

This section establishes the local existence and uniqueness of solutions to (21). More precisely, we prove the following proposition.

Proposition 2. *Consider the initial-value problem (21). Assume that the initial data* $(v_{0h}, \widetilde{w}_0, \widetilde{\rho}_0) \in H^{0,1}$ *satisfies* $\nabla_h \cdot v_{0h} + \partial_3 \widetilde{w}_0 = 0$*. Then, there is* $T_0 > 0$ *and a unique solution* $(v_h, \widetilde{w}, \widetilde{\rho})$ *of (21) on* $([0, T_0)$ *satisfying*

$$(v_h, \widetilde{w}, \widetilde{\rho}) \in C([0, T_0); L^2), \quad (\nabla_h v_h, \nabla_h \widetilde{w}, \nabla_h \widetilde{\rho}) \in L^2([0, T_0); H^{0,1}). \tag{64}$$

Proof. Again for notational convenience, we omit tildes and write (v_h, w, ρ) for $(v_h, \widetilde{w}, \widetilde{\rho})$. The local existence result is shown via Friedrichs' method. We introduce a few notation. For $n \in \mathbb{N}^+$, we define

$$\mathbb{E}_n f = \mathcal{F}^{-1}(\chi_{B(0,n)} \widehat{f}), \tag{65}$$

$$L_n^2 = \left\{ f \in L^2(\mathbb{R}^3) | \operatorname{supp} \widehat{f} \subset B(0, n) \right\}, \tag{66}$$

$$L_n^{2,\sigma} = \left\{ u \in L^2(\mathbb{R}^3) | \nabla \cdot u = 0, \operatorname{supp} \widehat{u} \subset B(0, n) \right\}, \tag{67}$$

where $\widehat{f}$ and $\mathcal{F}^{-1} f$ denote the Fourier transform and the inverse Fourier transform, respectively, and $\chi_{B(0,n)}$ denotes the characteristic function on the ball $B(0, n)$. Both L_n^2 and $L_n^{2,\sigma}$ are equipped with the L^2-norm.

Due to the divergence-free condition, the pressure term can be represented in terms of (v_h, w, ρ). By taking the divergence of the velocity equation in (21), we obtain

$$-\Delta p = \nabla_h \cdot P_1 + \partial_3 P_2, \tag{68}$$

where

$$P_1 = v_h \cdot \nabla_h v_h + w \partial_3 v_h + w^{(0)} \partial_3 v_h + f v_h^{\perp}, \tag{69}$$

$$P_2 = v_h \cdot \nabla_h w + w \partial_3 w + v_h \cdot \nabla_h w^{(0)} + w^{(0)} \partial_3 w + \rho. \tag{70}$$

Therefore,

$$p = p(v_h, w, \rho) := (-\Delta)^{-1} (\nabla_h \cdot P_1 + \partial_3 P_2). \tag{71}$$

To construct the solution of (21), we seek a sequence of solutions

$$\left\{(v_h^{(n)}, w^{(n)}, \rho^{(n)})\right\}_{n=1}^{\infty} \quad \text{with} \quad (v_h^{(n)}, w^{(n)}) \in L_n^{2,\sigma}, \quad \rho^{(n)} \in L_n^2 \tag{72}$$

to the following regularized system:

$$\begin{cases} \partial_t v_h^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + w^{(0)}\partial_3 v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)} + f(v_h^{(n)})^{\perp}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\nabla_h p^{(n)} + \nu\Delta_h v_h^{(n)}, \\ \partial_t w^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h w^{(0)} + v_h^{(n)} \cdot \nabla_h w^{(n)} + w^{(n)}\partial_3 w^{(n)} + w^{(0)}\partial_3 w^{(n)}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\partial_3 p^{(n)} - \rho^{(n)} + \nu\Delta_h w^{(n)}, \\ \nabla_h \cdot v_h^{(n)} + \partial_3 w^{(n)} = 0, \\ \partial_t \rho^{(n)} + \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h \rho^{(n)} + v_h^{(n)} \cdot \nabla_h \rho^{(0)} + w^{(n)}\partial_3 \rho^{(n)} + w^{(0)}\partial_3 \rho^{(n)}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad -w^{(n)} = \kappa\Delta_h \rho^{(n)}, \\ (v_h^{(n)}, w^{(n)}, \rho^{(n)})|_{t=0} = \mathbb{E}_n(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)}), \end{cases} \tag{73}$$

where $p^{(n)} = \mathbb{E}_n p(v_h^{(n)}, w^{(n)}, \rho^{(n)})$ with p as defined in (71). The system in (73) can be written as an ordinary differential equation

$$\frac{d}{dt}(v_h^{(n)}, w^{(n)}, \rho^{(n)}) = F_n(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \tag{74}$$

on the Banach space

$$(v_h^{(n)}, w^{(n)}) \in L_n^{2,\sigma}, \quad \rho^{(n)} \in L_n^2. \tag{75}$$

Here, F_n denotes all other terms except those with the time derivative in the equations in (73). It is not difficult to verify that F_n maps $L_n^{2,\sigma} \times L_n^2$ to $L_n^{2,\sigma} \times L_n^2$, and is locally Lipschitz. We take the term $\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)})$ from F_n as an example on how to verify these properties. By Lemma A.1 on the Bernstein inequality for functions whose Fourier transforms have compact supports,

$$\begin{aligned} \|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)})\|_{L^2(\mathbb{R}^3)} &\le \|v_h^{(n)} \cdot \nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\ &\le \|v_h^{(n)}\|_{L^\infty(\mathbb{R}^3)} \|\nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\ &\le C\, n^{\frac{3}{2}} \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)}\, n \|v_h^{(n)}\|_{L^2(\mathbb{R}^3)} \\ &= C\, n^{\frac{5}{2}} \, \|v_h^{(n)}\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{76}$$

Similarly,

$$\begin{aligned} &\|\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)}) - \mathbb{E}_n(u_h^{(n)} \cdot \nabla_h u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \\ &\le \|(v_h^{(n)} - u_h^{(n)}) \cdot \nabla_h v_h^{(n)}\|_{L^2(\mathbb{R}^3)} + \|u_h^{(n)} \cdot (\nabla_h v_h^{(n)} - \nabla_h u_h^{(n)})\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

$$\leq C n^{\frac{5}{2}} \left(\|v_h^{(n)})\|^2_{L^2(\mathbb{R}^3)} + \|u_h^{(n)}\|^2_{L^2(\mathbb{R}^3)} \right) \|v_h^{(n)} - u_h^{(n)}\|^2_{L^2(\mathbb{R}^3)}. \tag{77}$$

The local Lipschitz properties for other terms can be similarly verified. It then follows from the local existence and uniqueness theory on the ODEs in Banach spaces (see Theorem A.1) that, for any $n \in \mathbb{N}$, there is $T_n > 0$ and a unique solution $(v_h^{(n)}, w^{(n)}, \rho^{(n)})$ of (73) satisfying

$$(v_h^{(n)}, w^{(n)}\rho^{(n)}) \in C^\infty([0, T_n); L_n^{2,\sigma}) \times C^\infty([0, T_n); \times L_n^2). \tag{78}$$

Since the solution is infinitely smooth in both x and t, we can perform the same L^2-estimate as in (35) and obtain the same bound as in (40),

$$\begin{aligned} &\frac{d}{dt}\|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|^2_{L^2} + 2\nu\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)}, \nabla_h \rho^{(n)})\|^2_{L^2} \\ &\leq C(\nu^{-1}\|\nabla_h w^{(0)}\|^2_{L^2(\mathbb{R}^2)} + \nu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}\|\nabla_h \rho^{(0)}\|^2_{L^2(\mathbb{R}^2)})\|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|^2_{L^2}. \end{aligned} \tag{79}$$

Gronwall's inequality then implies the global upper bound

$$\begin{aligned} \|(v_h^{(n)}, w^{(n)}, \rho^{(n)})(t)\|_{L^2} &\leq K_0\|(v_h^{(n)}, w^{(n)}, \rho^{(n)})(0)\|_{L^2} \\ &\leq K_0\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}. \end{aligned} \tag{80}$$

By the extension theorem (see Theorem A.2), the solution $(v_h^{(n)}, w^{(n)}, \rho^{(n)})$ is global in time and

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \in C^\infty([0, \infty); L_n^{2,\sigma}) \times C^\infty([0, \infty); \times L_n^2). \tag{81}$$

Next we show that there is $T > 0$ independent of n such that

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \in L^\infty(0, T; H^{0,1}) \cap L^2(0, T; H^{1,1}) \tag{82}$$

with its norm bounded uniformly in this space. As in the proof of (55), we can show that

$$\begin{aligned} &\frac{d}{dt}\|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|^2_{H^{0,1}} + \nu\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|^2_{H^{0,1}} + \kappa\|\nabla_h \rho^{(n)}\|^2_{H^{0,1}} \\ &\leq C\left(\nu^{-1}\|\nabla_h w^{(0)}\|^2_{L^2(\mathbb{R}^2)} + \nu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}\|\nabla_h \rho^{(0)}\|^2_{L^2(\mathbb{R}^2)}\right)\|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|^2_{H^{0,1}} \\ &\quad + C\Big(\nu^{-3}\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|^2_{L^2} + \nu^{-1}\kappa^{-2}\|(\nabla_h v_h^{(n)}, \nabla_h \rho^{(n)})\|^2_{L^2}\Big) \\ &\quad \times \|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|^4_{H^{0,1}}. \end{aligned} \tag{83}$$

The proof of (83) is omitted due to the similarities with the proof of (55). Writing

$$a(t) := C\left(\nu^{-1}\|\nabla_h w^{(0)}\|^2_{L^2(\mathbb{R}^2)} + \nu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}\|\nabla_h \rho^{(0)}\|^2_{L^2(\mathbb{R}^2)}\right),$$

$$A(t) := \mathrm{e}^{-\int_0^t a(\tau)\,d\tau}\, \|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|_{H^{0,1}}^2, \tag{84}$$

we convert (83) into the inequality

$$\begin{aligned}\frac{d}{dt}A(t) &\le C\left(\nu^{-3}\,\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|_{L^2}^2 + \nu^{-1}\,\kappa^{-2}\,\|(\nabla_h v_h^{(n)}, \nabla_h \rho^{(n)})\|_{L^2}^2\right)\\ &\quad\times \mathrm{e}^{\int_0^t a(\tau)\,d\tau}\, A^2(t)\\ &\le CK_0\, A^2(t)\Big(\nu^{-3}\,\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|_{L^2}^2 + \nu^{-1}\,\kappa^{-2}\,\|(\nabla_h v_h^{(n)}, \nabla_h \rho^{(n)})\|_{L^2}^2\Big),\end{aligned} \tag{85}$$

where K_0 is as defined in (30). Integrating in time yields

$$-\frac{1}{A(t)} + \frac{1}{A(0)} = L(t), \tag{86}$$

where

$$L(t) = C\,K_0 \int_0^t \left(\nu^{-3}\,\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|_{L^2}^2 + \nu^{-1}\,\kappa^{-2}\,\|(\nabla_h v_h^{(n)}, \nabla_h \rho^{(n)})\|_{L^2}^2\right) d\tau. \tag{87}$$

Our next big step is to estimate $L(t)$. To do so, we choose a sufficiently large integer n_0 and define

$$v_F = \mathrm{e}^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(v_{0h}), \quad w_F = \mathrm{e}^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(w_0 - w_0^{(0)}), \quad \rho_F = \mathrm{e}^{\kappa\Delta_h t}\,\mathbb{E}_{n_0}(\rho_0 - \rho_0^{(0)}). \tag{88}$$

We remark that n_0 is fixed and independent of n. n_0 is taken to be large to make sure that part of the upper bound in (116), namely,

$$\|(I - \mathbb{E}_{n_0})(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}^2 \tag{89}$$

is small. n is an arbitrary integer. Since our focus is mainly on large n, we can assume without loss of generality that $n \ge n_0$. In addition, it is easy to check that

$$\nabla_h \cdot v_F + \partial_3 w_F = 0. \tag{90}$$

In fact, the divergence-free condition on the initial data $\nabla_h \cdot v_{0h} + \partial_3 w_0 = 0$ and the fact that $\partial_3 w_0^{(0))} = 0$ since $w_0^{(0))}(x_h)$ depends only on x_h,

$$\begin{aligned}\nabla_h \cdot v_F + \partial_3 w_F &= \nabla_h \cdot e^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(v_{0h}) + \partial_3 e^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(w_0 - w_0^{(0)})\\ &= e^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(\nabla_h \cdot v_{0h}) + e^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(\partial_3(w_0 - w_0^{(0)}))\\ &= e^{\nu\Delta_h t}\,\mathbb{E}_{n_0}(\nabla_h \cdot v_{0h} + \partial_3 w_0) = 0.\end{aligned} \tag{91}$$

We now split each of $v_h^{(n)}$, $w^{(n)}$, and $\rho^{(n)}$ into two parts:

$$v_h^{(n)} = u_h^{(n)} + v_F, \quad w^{(n)} = q^{(n)} + w_F, \quad \rho^{(n)} = \theta^{(n)} + \rho_F. \tag{92}$$

Then, $(u_h^{(n)}, q^{(n)}, \theta^{(n)})$ with $n \geq n_0$ satisfies

$$\begin{cases} \partial_t u_h^{(n)} + \mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h v_h^{(n)} + w^{(0)}\partial_3 v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)} + f(v_h^{(n)})^{\perp}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\nabla_h p^{(n)} + \nu\Delta_h u_h^{(n)}, \\ \partial_t q^{(n)} + \mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h w^{(0)} + v_h^{(n)}\cdot\nabla_h w^{(n)} + w^{(n)}\partial_3 w^{(n)} + w^{(0)}\partial_3 w^{(n)}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\partial_3 p^{(n)} - \rho^{(n)} + \nu\Delta_h q^{(n)}, \\ \nabla_h\cdot v_h^{(n)} + \partial_3 w^{(n)} = 0, \\ \partial_t\theta^{(n)} + \mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h\rho^{(n)} + v_h^{(n)}\cdot\nabla_h\rho^{(0)} + w^{(n)}\partial_3\rho^{(n)} + w^{(0)}\partial_3\rho^{(n)}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad -w^{(n)} = \kappa\Delta_h\theta^{(n)}, \\ (u_h^{(n)}, q^{(n)}, \theta^{(n)})|_{t=0} = (\mathbb{E}_n - \mathbb{E}_{n_0})(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)}). \end{cases} \tag{93}$$

Testing the equations in (93) with $(u_h^{(n)}, q^{(n)}, \theta^{(n)})$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u_h^{(n)}, q^{(n)}, \theta^{(n)})\|_{L^2}^2 + \nu\|(\nabla_h u_h^{(n)}, \nabla_h q^{(n)})\|_{L^2}^2 + \kappa\|\nabla_h\theta^{(n)}\|_{L^2}^2 \\ &= -\int u_h^{(n)}\cdot\mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h v_h^{(n)} + w^{(0)}\partial_3 v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)} + f(v_h^{(n)})^{\perp}\Big)\,dx \\ &\quad - \int q^{(n)}\cdot\mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h w^{(0)} + v_h^{(n)}\cdot\nabla_h w^{(n)} + w^{(n)}\partial_3 w^{(n)} + w^{(0)}\partial_3 w^{(n)}\Big)\,dx \\ &\quad - \int \theta^{(n)}\cdot\mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h\rho^{(n)} + v_h^{(n)}\cdot\nabla_h\rho^{(0)} + w^{(n)}\partial_3\rho^{(n)} + w^{(0)}\partial_3\rho^{(n)}\Big)\,dx \\ &\quad - \int(\rho^{(n)}\,q^{(n)} - w^{(n)}\,\theta^{(n)})\,dx, \end{aligned} \tag{94}$$

where we have used the divergence-free condition $\nabla_h\cdot u_h^{(n)} + \partial_3 q^{(n)} = 0$ to eliminate the pressure term. In fact,

$$-\int(u_h^{(n)}\cdot\nabla_h p^{(n)} + q^{(n)}\partial_3 p^{(n)})\,dx = \int(\nabla_h\cdot u_h^{(n)} + \partial_3 q^{(n)})p^{(n)}\,dx = 0 \tag{95}$$

due to $\nabla_h\cdot u_h^{(n)} + \partial_3 q^{(n)} = 0$. To estimate the terms on the right-hand side, we first notice that

$$\begin{aligned} \|u_h^{(n)}\|_{L^2} &\leq \|v_h^{(n)}\|_{L^2} + \|v_F\|_{L^2} \\ &\leq K_0\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2} + \|v_{0h}\|_{L^2} \leq M_0, \end{aligned} \tag{96}$$

where

$$M_0 := (K_0 + 1)\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}. \tag{97}$$

Similarly,

$$\|q^{(n)}\|_{L^2} \leq M_0, \quad \|\theta^{(n)}\|_{L^2} \leq M_0. \tag{98}$$

Without loss of generality, we assume $n \geq n_0$. Clearly,

$$\mathbb{E}_n u_h^{(n)} = u_h^{(n)}, \quad \mathbb{E}_n q^{(n)} = q^{(n)}, \quad \mathbb{E}_n \theta^{(n)} = \theta^{(n)}. \tag{99}$$

Therefore, by the fact that $\mathbb{E}_n$ is self-adjoint, we have

$$\begin{aligned}
& -\int u_h^{(n)} \cdot \mathbb{E}_n\Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + w^{(0)}\partial_3 v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)} + f(v_h^{(n)})^{\perp}\Big)\,dx \\
&= -\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + w^{(0)}\partial_3 v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)} + f(v_h^{(n)})^{\perp}\Big)\,dx \\
&= -\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h u_h^{(n)} + w^{(n)}\,\partial_3 u_h^{(n)}\Big) + u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_F + w^{(n)}\,\partial_3 v_F\Big)dx \\
&\quad - \int u_h^{(n)} \cdot \Big(w^{(0)}\partial_3 u_h^{(n)} + w^{(0)}\partial_3 v_F + f(u_h^{(n)})^{\perp} + f v_F^{\perp}\Big)\,dx \\
&= -\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h v_F + w^{(n)}\,\partial_3 v_F\Big)dx - \int u_h^{(n)} \cdot \big(w^{(0)}\partial_3 v_F + f v_F^{\perp}\big)\,dx \\
&\leq \|u_h^{(n)}\|_{L^2}\|v_h^{(n)}\|_{L^2}\|\nabla_h v_F\|_{L^\infty} + \|u_h^{(n)}\|_{L^2}\|w^{(n)}\|_{L^2}\|\partial_3 v_F\|_{L^\infty} \\
&\quad + \|u_h^{(n)}\|_{L^2}\|w^{(0)}\|_{L^2(\mathbb{R}^2)}\|\partial_3 v_F\|_{L^2_{x_3}L^\infty_h} + C\,\|u_h^{(n)}\|_{L^2}\|v_F\|_{L^2} \\
&\leq C\,n_0^{\frac{5}{2}} M_0^3 + C\,n_0^2 M_0^2 \|w^{(0)}\|_{L^2(\mathbb{R}^2)} + C\,M_0^2.
\end{aligned} \tag{100}$$

Here, we have used the following facts, due to $\nabla_h \cdot v_h^{(n)} + \partial_3 w^{(n)} = 0$ and $w^{(0)}$ independent of x_3,

$$\int u_h^{(n)} \cdot \Big(v_h^{(n)} \cdot \nabla_h u_h^{(n)} + w^{(n)}\,\partial_3 u_h^{(n)}\Big)\,dx = 0, \tag{101}$$

$$\int u_h^{(n)} \cdot w^{(0)}\partial_3 u_h^{(n)}\,dx = 0, \quad u_h^{(n)} \cdot (u_h^{(n)})^{\perp} = 0. \tag{102}$$

In addition, we have used Lemma A.1 to obtain the bounds

$$\|\nabla_h v_F\|_{L^\infty} \leq C\,n_0^{\frac{5}{2}}\|v_{0h}\|_{L^2}, \quad \|\partial_3 v_F\|_{L^2_{x_3}L^\infty_h} \leq C\,n_0^2\|v_{0h}\|_{L^2}. \tag{103}$$

The upper bound obtained here depends on n_0 and the initial data only. It is independent of n. This point is crucial to obtain a time interval independent of n. Many other terms in (94) can be similarly estimated. There are two terms that need a slightly different treatment,

$$\int q^{(n)} \cdot \mathbb{E}_n(v_h^{(n)} \cdot \nabla_h w^{(0)})\,dx, \qquad \int \theta^{(n)} \cdot \mathbb{E}_n(v_h^{(n)} \cdot \nabla_h \rho^{(0)})\,dx. \tag{104}$$

By (99), we have

$$\begin{aligned}
&\int q^{(n)} \cdot \mathbb{E}_n(v_h^{(n)} \cdot \nabla_h w^{(0)})\,dx \\
&= \int q^{(n)} \cdot (v_h^{(n)} \cdot \nabla_h w^{(0)})\,dx \\
&= \int q^{(n)} \cdot (u_h^{(n)} \cdot \nabla_h w^{(0)} + v_F \cdot \nabla_h w^{(0)})\,dx \\
&= \int \Big(q^{(n)} \cdot (u_h^{(n)} \cdot \nabla_h w^{(0)}) - v_F \cdot \nabla_h q^{(n)}\, w^{(0)} - q^{(n)} \nabla_h \cdot v_F w^{(0)}\Big)\,dx \\
&\le \|q^{(n)}\|_{L^2_{x_3}L^4_h} \|u_h^{(n)}\|_{L^2_{x_3}L^4_h} \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\quad + \|\nabla_h q^{(n)}\|_{L^2} \|v_F\|_{L^2_{x_3}L^\infty_h} \|w^{(0)}\|_{L^2(\mathbb{R}^2)} + \|q^{(n)}\|_{L^2} \|\nabla_h \cdot v_F\|_{L^2_{x_3}L^\infty_h} \|w^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\le C\,\|q^{(n)}\|_{L^2}^{\frac{1}{2}} \|\nabla_h q^{(n)}\|_{L^2}^{\frac{1}{2}} \|u_h^{(n)}\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_h^{(n)}\|_{L^2}^{\frac{1}{2}} \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\quad + C\, n_0 M_0\, \|\nabla_h q^{(n)}\|_{L^2} \|w^{(0)}\|_{L^2(\mathbb{R}^2)} + C\, n_0^2 M_0^2 \|w^{(0)}\|_{L^2(\mathbb{R}^2)} \\
&\le \frac{\nu}{4}\Big(\|\nabla_h q^{(n)}\|_{L^2}^2 + \|\nabla_h u_h^{(n)}\|_{L^2}^2\Big) \\
&\quad + C\,\nu^{-1} \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 (\|q^{(n)}\|_{L^2}^2 + \|u_h^{(n)}\|_{L^2}^2) \\
&\quad + C\, n_0^2 M_0^2 \,(\|w^{(0)}\|_{L^2(\mathbb{R}^2)} + \nu^{-1} \|w^{(0)}\|_{L^2(\mathbb{R}^2)}^2).
\end{aligned} \tag{105}$$

The other term in (104) can be similarly bounded,

$$\begin{aligned}
&\int \theta^{(n)} \cdot \mathbb{E}_n(v_h^{(n)} \cdot \nabla_h \rho^{(0)})\,dx \\
&\le \frac{\nu}{4} \|\nabla_h u^{(n)}\|_{L^2}^2 + \frac{\kappa}{4} \|\nabla_h \theta^{(n)}\|_{L^2}^2 \\
&\quad + C\,\nu^{-\frac{1}{2}} \kappa^{-\frac{1}{2}} \|\nabla_h \rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2 (\|u_h^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2) \\
&\quad + C\, n_0^2 M_0^2 \,(\|\rho^{(0)}\|_{L^2(\mathbb{R}^2)} + \kappa^{-1} \|\rho^{(0)}\|_{L^2(\mathbb{R}^2)}^2).
\end{aligned} \tag{106}$$

Invoking these estimates leads to

$$\begin{aligned}
&\frac{d}{dt} \|(u_h^{(n)}, q^{(n)}, \theta^{(n)})\|_{L^2}^2 + \nu \|(\nabla_h u_h^{(n)}, \nabla_h q^{(n)})\|_{L^2}^2 + \kappa \|\nabla_h \theta^{(n)}\|_{L^2}^2 \\
&\le C\,(\nu^{-1} + \nu^{-\frac{1}{2}} \kappa^{-\frac{1}{2}}) \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \|(u_h^{(n)}, q^{(n)}, \theta^{(n)})\|_{L^2}^2 + Q_0,
\end{aligned} \tag{107}$$

where Q_0 is given by

$$Q_0 := C\, n_0^{\frac{5}{2}} M_0^3 + C\, n_0^2 M_0^2 (\|w^{(0)}\|_{L^2(\mathbb{R}^2)} + (\nu^{-1} + \kappa^{-1}) \|w^{(0)}\|_{L^2(\mathbb{R}^2)}^2) + C\, M_0^2. \tag{108}$$

It then follows from Gronwall's inequality that

$$\begin{aligned}
&\|(u_h^{(n)}, q^{(n)}, \theta^{(n)})(t)\|_{L^2}^2 + \nu \int_0^t \|(\nabla_h u_h^{(n)}, \nabla_h q^{(n)})\|_{L^2}^2 \, d\tau + \kappa \int_0^t \|\nabla_h \theta^{(n)}\|_{L^2}^2 \, d\tau \\
&\le \mathrm{e}^{C(\nu^{-1} + \nu^{-\frac{1}{2}} \kappa^{-\frac{1}{2}}) \int_0^t \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \, d\tau} \left(\|(u_h^{(n)}, q^{(n)}, \theta^{(n)})(0)\|_{L^2}^2 + Q_0 t \right) \\
&\le C\, K_0 \|(I - \mathbb{E}_{n_0})(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}^2 + C\, K_0 Q_0 t,
\end{aligned} \tag{109}$$

where we have used the following simple facts:

$$\mathrm{e}^{C(\nu^{-1} + \nu^{-\frac{1}{2}} \kappa^{-\frac{1}{2}}) \int_0^t \|\nabla_h w^{(0)}\|_{L^2(\mathbb{R}^2)}^2 \, d\tau} \le C\, K_0 \tag{110}$$

and

$$\begin{aligned}
&\|(u_h^{(n)}, q^{(n)}, \theta^{(n)})(0)\|_{L^2} \\
&= \|(v_h^{(n)}(0) - v_F(0), w^{(n)}(0) - w_F(0), \rho^{(n)}(0) - \rho_F(0))\|_{L^2} \\
&\le \|(I - \mathbb{E}_{n_0})(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}.
\end{aligned} \tag{111}$$

We are now ready to estimate the term on the right-hand side of (86), namely,

$$L(t) := C\, K_0 \int_0^t \left(\nu^{-3} \|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|_{L^2}^2 + \nu^{-1} \kappa^{-2} \|(\nabla_h v_h^{(n)}, \nabla_h \rho^{(n)})\|_{L^2}^2 \right) d\tau. \tag{112}$$

We have written

$$v_h^{(n)} = u_h^{(n)} + v_F, \quad w^{(n)} = q^{(n)} + w_F, \quad \rho^{(n)} = \theta^{(n)} + \rho_F. \tag{113}$$

Clearly, by the definition of $v_F = e^{\nu \Delta_h t}\, \mathbb{E}_{n_0}(v_{0h})$,

$$\int_0^t \|\nabla_h v_F\|_{L^2}^2 \, d\tau \le n_0^2 \int_0^t \|\mathbb{E}_{n_0}(v_{0h})\|_{L^2}^2 \, d\tau \le n_0^2 \|v_{0h}\|_{L^2}^2 \, t. \tag{114}$$

Similarly,

$$\int_0^t \|\nabla_h w_F\|_{L^2}^2 \, d\tau \le n_0^2 \|w_0 - w_0^{(0)}\|_{L^2}^2 \, t, \quad \int_0^t \|\nabla_h \rho_F\|_{L^2}^2 \, d\tau \le n_0^2 \|\rho_0 - \rho_0^{(0)}\|_{L^2}^2 \, t. \tag{115}$$

Therefore, by (109), we can bound $L(t)$ by

$$
\begin{aligned}
L(t) &:= C\,K_0 \int_0^t \left(\nu^{-3}\,\|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|_{L^2}^2 + \nu^{-1}\,\kappa^{-2}\,\|(\nabla_h v_h^{(n)}, \nabla_h \rho^{(n)})\|_{L^2}^2\right) d\tau \\
&\le CK_0(\nu^{-3} + \nu^{-1}\,\kappa^{-2})\Bigg(\int_0^t \|(\nabla_h u_h^{(n)}, \nabla_h q^{(n)}, \nabla_h \theta^{(n)})\|_{L^2}^2\, d\tau \\
&\qquad + \int_0^t \|(\nabla_h v_F, \nabla_h w_F, \nabla_h \rho_F)\|_{L^2}^2\, d\tau\Bigg) \\
&\le CK_0^2(\nu^{-4} + \nu^{-1}\,\kappa^{-3})\,\|(I - \mathbb{E}_{n_0})(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}^2 \\
&\quad + CK_0(\nu^{-3} + \nu^{-1}\,\kappa^{-2})n_0^2\Big(\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{L^2}^2 + K_0 Q_0\Big)t.
\end{aligned} \tag{116}
$$

To obtain an upper bound for $A(t)$ in (86), we recall (84) to get

$$
A(0) = \|\mathbb{E}_n(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{H^{0,1}}^2. \tag{117}
$$

If we choose n_0 sufficiently large and $t \le T = T(n_0)$ for sufficiently small $T > 0$, then the upper bound for $L(t)$ in (116) can be made sufficiently small so that

$$
1 - L(t)A(0) \ge \frac{1}{2}, \quad 0 < t \le T. \tag{118}
$$

It then follows from (86) that, for any $0 < t \le T$,

$$
\frac{1}{A(t)} = \frac{1 - L(t)A(0)}{A(0)} \ge \frac{1}{2A(0)} \tag{119}
$$

or

$$
A(t) \le 2A(0) \le 2\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{H^{0,1}}^2. \tag{120}
$$

By the definition of $A(t)$ in (84),

$$
\|(v_h^{(n)}, w^{(n)}, \rho^{(n)})\|_{H^{0,1}}^2 = e^{\int_0^t a(\tau)\,d\tau}\,A(t) \le 2K_0\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{H^{0,1}}^2. \tag{121}
$$

Integrating (83) in time yields the upper bound, for any $t \in [0, T]$,

$$
\begin{aligned}
&\nu \int_0^t \|(\nabla_h v_h^{(n)}, \nabla_h w^{(n)})\|_{H^{0,1}}^2\, d\tau + \kappa \int_0^t \|\nabla_h \rho^{(n)}\|_{H^{0,1}}^2\, d\tau \\
&\le C\,\|(v_{0h}, w_0 - w_0^{(0)}, \rho_0 - \rho_0^{(0)})\|_{H^{0,1}}^2.
\end{aligned} \tag{122}
$$

Thus, we have shown that there is $T > 0$ independent of n such that

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \in L^\infty(0,T;H^{0,1}) \cap L^2(0,T;H^{1,1}) \tag{123}$$

with its norm in this space bounded uniformly in terms of n.

The next step is to show that $(v_h^{(n)}, w^{(n)}, \rho^{(n)})$ has a convergent subsequence whose limit solves (11). By Banach–Alaoglu theorem, there is a subsequence, still denoted by $(v_h^{(n)}, w^{(n)}, \rho^{(n)})$, and $(v_h, w, \rho) \in L^\infty(0,T;H^{0,1}) \cap L^2(0,T;H^{1,1})$ satisfying

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \rightharpoonup (v_h, w, \rho) \quad \text{in } H^{0,1} \text{ for almost every } t, \tag{124}$$

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \rightharpoonup (v_h, w, \rho) \quad \text{in } L^2(0,T;H^{1,1}). \tag{125}$$

These weak convergences here are not sufficient to show that (v_h, w, ρ) solves the Boussinesq system in (11). We need to obtain some strong convergence. We will show that

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \to (v_h, w, \rho) \quad \text{in } L^2(0,T;L^2). \tag{126}$$

The tool is the Aubin–Lions–Simon lemma stated below.

Lemma 2 (Aubin–Lions–Simon). *Let X_0, X, and X_1 be three Banach spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$. Suppose X_0 is compactly embedded in X and X is continuously embedded in X_1. Let $1 \le p, q \le \infty$. Set*

$$W = \{f \in L^p(0,T;X_0) |\, \partial_t f \in L^q(0,T;X_1)\}. \tag{127}$$

(1) If $p < \infty$, then the embedding of W into $L^p(0,T;X)$ is compact,
(2) if $p = +\infty$ and $q > 1$, then the embedding of W into $C([0,T];X)$ is compact.

To apply this lemma, we need to show that

$$(\partial_t v_h^{(n)}, \partial_t w^{(n)}, \partial_t \rho^{(n)}) \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{128}$$

As our first step, we show that

$$(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \in L^4(0,T;L^4(\mathbb{R}^3)). \tag{129}$$

Due to the 2D interpolation inequality

$$\|f\|_{L^4(0,T;\dot{H}^{\frac{1}{2}}(\mathbb{R}^2))} \le C\, \|f\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\mathbb{R}^2))} \|f\|^{\frac{1}{2}}_{L^2(0,T;\dot{H}^1(\mathbb{R}^2))}, \tag{130}$$

we have

$$L^\infty(0,T;L^2(\mathbb{R}^2))\cap L^2(0,T;\dot H^1(\mathbb{R}^2))\hookrightarrow L^4(0,T;\dot H^{\frac12}(\mathbb{R}^2)). \tag{131}$$

As a consequence,

$$L^\infty(0,T;H^{0,1}(\mathbb{R}^3))\cap L^2(0,T;H^{1,1}(\mathbb{R}^3))\hookrightarrow L^4(0,T;H^{\frac12,1}(\mathbb{R}^3)). \tag{132}$$

Therefore,

$$(v_h^{(n)},w^{(n)},\rho^{(n)})\in L^4(0,T;H^{\frac12,1}(\mathbb{R}^3)). \tag{133}$$

Furthermore, we have the embedding

$$H^{\frac12,1}(\mathbb{R}^3)\hookrightarrow\ L^2_{x_3}L^4_h(\mathbb{R}^3)\cap L^\infty_{x_3}L^4_h(\mathbb{R}^3) \tag{134}$$

because of the embedding inequalities

$$\|f\|_{L^2_{x_3}L^4_h}\le C\,\|f\|_{L^2_{x_3}H_h^{\frac12}}\le C\,\|f\|_{H^1_{x_3}H_h^{\frac12}}=C\,\|f\|_{H^{\frac12,1}} \tag{135}$$

and

$$\|f\|_{L^\infty_{x_3}L^4_h}\le C\,\|f\|_{L^\infty_{x_3}H_h^{\frac12}}\le C\,\|f\|_{H^1_{x_3}H_h^{\frac12}}=C\,\|f\|_{H^{\frac12,1}}. \tag{136}$$

Therefore,

$$L^4(0,T;H^{\frac12,1})\hookrightarrow L^4(0,T;L^2_{x_3}L^4_h\cap L^\infty_{x_3}L^4_h)\hookrightarrow L^4(0,T;L^4) \tag{137}$$

and

$$(v_h^{(n)},w^{(n)},\rho^{(n)})\in L^4(0,T;L^4(\mathbb{R}^3)). \tag{138}$$

Now we are ready to show that

$$\partial_t v_h^{(n)}\in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{139}$$

We check the terms in the equation of $v_h^{(n)}$,

$$\partial_t v_h^{(n)}+\mathbb{E}_n\Big(v_h^{(n)}\cdot\nabla_h v_h^{(n)}+w^{(0)}\partial_3 v_h^{(n)}+w^{(n)}\,\partial_3 v_h^{(n)}+f(v_h^{(n)})^\perp\Big)$$
$$=-\nabla_h p^{(n)}+\nu\Delta_h v_h^{(n)}. \tag{140}$$

Since $v_h^{(n)} \in L^2(0,T;H^{1,1}(\mathbb{R}^3))$, we have

$$\Delta_h v_h^{(n)} \in L^2(0,T;H^{-1,1}(\mathbb{R}^3)) \hookrightarrow L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{141}$$

Due to $\nabla_h \cdot v_h^n + \partial_3 w^{(n)} = 0$,

$$\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)}) = \nabla_h \cdot \mathbb{E}_n(v_h^{(n)} \otimes v_h^{(n)}) + \partial_3 \mathbb{E}_n(w^{(n)}\,v_h^{(n)}) \tag{142}$$

and $(v_h^{(n)}, w^{(n)}, \rho^{(n)}) \in L^4(0,T;L^4(\mathbb{R}^3))$ and Hölder's inequality imply

$$\mathbb{E}_n(v_h^{(n)} \otimes v_h^{(n)}), \quad \mathbb{E}_n(w^{(n)}\,v_h^{(n)}) \in L^2(0,T;L^2). \tag{143}$$

Therefore,

$$\mathbb{E}_n(v_h^{(n)} \cdot \nabla_h v_h^{(n)} + w^{(n)}\,\partial_3 v_h^{(n)}) \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{144}$$

Since $w^{(0)}$ is independent of x_3,

$$\mathbb{E}_n(w^{(0)}\partial_3 v_h^{(n)}) = \partial_3 \mathbb{E}_n(w^{(0)} v_h^{(n)}). \tag{145}$$

It is easy to check that $\mathbb{E}_n(w^{(0)} v_h^{(n)}) \in L^2(0,T;L^2(\mathbb{R}^3))$. In fact,

$$\begin{aligned}
&\|\mathbb{E}_n(w^{(0)} v_h^{(n)})\|_{L^2(0,T;L^2(\mathbb{R}^3))} \le \|\|w^{(0)}\|_{L^4(\mathbb{R}^2)}\,\|v_h^{(n)}\|_{L^2_{x_3}L^4_h}\|_{L^2(0,T)} \\
&\le \|w^{(0)}\|_{L^4(0,T;L^4(\mathbb{R}^2))}\,\|v_h^{(n)}\|_{L^4(0,T;L^2_{x_3}L^4_h)} \\
&\le C\,\|w^{(0)}\|_{L^4(0,T;H^{\frac{1}{2}}(\mathbb{R}^2))}\,\|v_h^{(n)}\|_{L^4(0,T;H^{\frac{1}{2},1})} \\
&\le C\,\|w^{(0)}\|_{L^\infty(0,T;L^2)\cap L^2(0,T;\dot{H}^1)}\,\|v_h^{(n)}\|_{L^4(0,T;H^{\frac{1}{2},1})} < \infty.
\end{aligned} \tag{146}$$

Therefore,

$$\mathbb{E}_n(w^{(0)}\partial_3 v_h^{(n)}) \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{147}$$

Since f is a smooth function with compact support,

$$\mathbb{E}_n(f(v_h^{(n)})^\perp) \in L^2(0,T;L^2) \hookrightarrow L^2(0,T;H^{-1}). \tag{148}$$

In addition, it is easy to check that

$$p^{(n)} \in L^2(0,T;L^2(\mathbb{R}^3)). \tag{149}$$

Therefore, $\nabla p \in L^2(0,T;H^{-1})$. Thus, we have shown

$$\partial_t v_h^{(n)} \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{150}$$

Similarly we can also show that

$$(\partial_t w^{(n)}, \partial_t \rho^{(n)}) \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{151}$$

To apply the Aubin–Lions–Simon lemma, we take the following spaces:

$$H^{-1}(B(0,m)) \hookrightarrow L^2(B(0,m)) \hookrightarrow H^1(B(0,m)), \tag{152}$$

where $B(0,m)$ is the ball of radius m with $m > 0$ being an integer. The first embedding in (152) is compact and the second is certainly continuous. The Aubin–Lions–Simon lemma does not directly apply to the spaces

$$H^{-1}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3) \tag{153}$$

since then the embeddings are no longer compact. The Aubin–Lions–Simon lemma implies that, for each fixed positive integer m, there is a subsequence

$$(v_h^{(n_{m,k})}, w^{(n_{m,k})}, \rho^{(n_{m,k})}) \quad \to \quad (v_h, w, \rho) \in L^2(0,T;L^2(B(0,m))) \quad \text{as } k \to \infty. \tag{154}$$

The Cantor diagonal process then allows us to obtain the desired subsequence

$$(v_h^{(n_{m,m})}, w^{(n_{m,m})}, \rho^{(n_{m,m})}) \quad \to \quad (v_h, w, \rho) \in L^2(0,T;L^2(\mathbb{R}^3)) \quad \text{as } m \to \infty. \tag{155}$$

We can then show that (v_h, w, ρ) solves the Boussinesq equation in (21) by taking the limit in (73). Since this process is standard, we omit the details.

It is easy to show that $(v_h, w, \rho) \in C([0,\infty); L^2)$. We combine the facts that, for any $T > 0$,

$$u \in L^2(0,T;H^{1,1}) \hookrightarrow L^2(0,T;H^1) \tag{156}$$

and $\partial_t u \in L^2(0,T;H^{-1})$ with Lemma A.2 (see Ref. [57, p. 303]) to obtain $(v_h, w, \rho) \in C([0,\infty); L^2)$.

We prove the uniqueness. Assume that

$$(v_h, w, \rho),\ (U_h, Q, \Theta) \in L^\infty(0,T;H^{0,1}) \cap L^2(0,T;H^{1,1}) \tag{157}$$

are two solutions of (21). Consider the difference $(\delta v_h, \delta w, \delta \rho)$ with

$$\delta v_h = v_h - U_h, \quad \delta w = w - Q, \quad \delta \rho = \rho - \Theta. \tag{158}$$

It is easy to check that $(\delta v_h, \delta w, \delta\rho)$ satisfies

$$\begin{cases} \partial_t \delta v_h + \delta v_h \cdot \nabla_h v_h + U_h \cdot \nabla_h \delta v_h + w^{(0)} \partial_3 \delta v_h + \delta w\, \partial_3 v_h \\ \qquad\qquad + Q\partial_3 \delta v_h + f(\delta v_h)^{\perp} = -\nabla_h \delta p + \nu \Delta_h \delta v_h, \\ \partial_t \delta w + \delta v_h \cdot \nabla_h w^{(0)} + \delta v_h \cdot \nabla_h w + U_h \cdot \nabla_h \delta w + \delta w \partial_3 w \\ \qquad\qquad + Q\partial_3 \delta w + w^{(0)} \partial_3 \delta w = -\partial_3 \delta p - \delta\rho + \nu \Delta_h \delta w, \\ \nabla_h \cdot \delta v_h + \partial_3 \delta w = 0, \\ \partial_t \delta\rho + \delta v_h \cdot \nabla_h \rho + U_h \cdot \nabla_h \delta\rho + \delta w \partial_3 \rho + Q\partial_3 \delta\rho + \delta v_h \cdot \nabla_h \rho^{(0)} \\ \qquad\qquad + w^{(0)} \partial_3 \delta\rho - \delta w = \kappa \Delta_h \delta\rho, \\ (\delta v_h, \delta w, \delta\rho)|_{t=0} = 0, \end{cases} \tag{159}$$

where δp denotes the corresponding pressure difference. Dotting (159) with $(\delta v_h, \delta w, \delta\rho)$ yields

$$\frac{1}{2}\frac{d}{dt}\|(\delta v_h, \delta w, \delta\rho)\|_{L^2}^2 + \nu\|(\nabla_h \delta v_h, \nabla_h \delta w)\|_{L^2}^2 + \kappa\|\nabla_h \delta\rho\|_{L^2}^2 = K_1 + \cdots K_8, \tag{160}$$

where

$$\begin{aligned} K_1 &= -\int \delta v_h \cdot \nabla_h v_h \cdot \delta v_h\, dx, \quad K_2 = -\int \delta w\, \partial_3 v_h \cdot \delta v_h\, dx, \\ K_3 &= -\int \delta v_h \cdot \nabla_h w\, \delta w\, dx, \quad K_4 = -\int \delta w\, \partial_3 w\, \delta w\, dx, \\ K_5 &= -\int \delta v_h \cdot \nabla_h \rho\, \delta\rho\, dx, \quad K_6 = -\int \delta\rho\, \partial_3 w\, \delta\rho\, dx, \\ K_7 &= -\int \delta v_h \cdot \nabla_h w^{(0)}\, \delta w\, dx, \quad K_8 = -\int \delta v_h \cdot \nabla_h \rho^{(0)}\, \delta\rho\, dx. \end{aligned} \tag{161}$$

Here, we have used the facts, due to $\nabla \cdot (U_h, Q) = \nabla_h \cdot U_h + \partial_3 Q = 0$, that

$$\begin{aligned} &\int (U_h \cdot \nabla_h \delta v_h + Q\partial_3 \delta u_h) \cdot \delta v_h = 0, \\ &\int (U_h \cdot \nabla_h \delta w + Q\partial_3 \delta w) \cdot \delta w = 0, \\ &\int (U_h \cdot \nabla_h \delta\rho + Q\partial_3 \delta\rho) \cdot \delta\rho = 0. \end{aligned} \tag{162}$$

By Lemma 1,

$$\begin{aligned} |K_1| \le & C\, \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \delta v_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \delta v_h\|_{L^2}^{\frac{1}{2}} \\ \le & \frac{\nu}{8}\|\nabla_h \delta v_h\|_{L^2}^2 + C\, \|\nabla_h v_h\|_{L^2}\, \|\partial_3 \nabla_h v_h\|_{L^2}\, \|\delta v_h\|_{L^2}^2. \end{aligned} \tag{163}$$

By Lemma 1 and $\partial_3 \delta w = -\nabla_h \cdot \delta v_h$,

$$
\begin{aligned}
|K_2| \leq & C \|\delta w\|_{L^2}^{\frac{1}{2}} \|\partial_3 \delta w\|_{L^2}^{\frac{1}{2}} \|\partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \|\partial_2 \delta v_h\|_{L^2}^{\frac{1}{2}} \\
\leq & C \|\delta w\|_{L^2}^{\frac{1}{2}} \|\nabla_h \delta v_h\|_{L^2} \|\partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 v_h\|_{L^2}^{\frac{1}{2}} \|\delta v_h\|_{L^2}^{\frac{1}{2}} \\
\leq & \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + C \|\partial_3 v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2} \|(\delta v_h, \delta w)\|_{L^2}^2.
\end{aligned} \tag{164}
$$

The estimate of K_3 is similar to that for K_1,

$$
|K_3| \leq \frac{\nu}{8} \|(\nabla_h \delta v_h, \nabla_h \delta w)\|_{L^2}^2 + C \|\nabla_h w\|_{L^2} \|\partial_3 \nabla_h w\|_{L^2} \|(\delta v_h, \delta w)\|_{L^2}^2. \tag{165}
$$

K_4 can be bounded similarly as K_2,

$$
|K_4| \leq \frac{\nu}{8} \|(\nabla_h \delta v_h, \nabla_h \delta w)\|_{L^2}^2 + C \|\partial_3 w\|_{L^2} \|\partial_3 \nabla_h w\|_{L^2} \|(\delta v_h, \delta w)\|_{L^2}^2. \tag{166}
$$

The bound for K_5 is similar to that for K_1,

$$
|K_5| \leq \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + \frac{\kappa}{8} \|\nabla_h \delta \rho\|_{L^2}^2 + C \|\nabla_h \rho\|_{L^2} \|\partial_3 \nabla_h \rho\|_{L^2} \|(\delta v_h, \delta \rho)\|_{L^2}^2. \tag{167}
$$

K_6 can be bounded similarly as K_2,

$$
|K_6| \leq \frac{\kappa}{8} \|\nabla_h \delta \rho\|_{L^2}^2 + C \|\nabla_h v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2} \|\delta \rho\|_{L^2}^2. \tag{168}
$$

To estimate the last two terms, we notice that $\nabla_h w^{(0)}$ and $\nabla_h \rho^{(0)}$ are independent of x_3 and apply Hölder's inequality to obtain

$$
|K_7| \leq \frac{\nu}{8} \|(\nabla_h \delta v_h, \nabla_h \delta w)\|_{L^2}^2 + C \|\nabla_h w^{(0)}\|_{L^2}^2 \|(\delta v_h, \delta w)\|_{L^2}^2, \tag{169}
$$

$$
|K_8| \leq \frac{\nu}{8} \|\nabla_h \delta v_h\|_{L^2}^2 + \frac{\kappa}{8} \|\nabla_h \delta \rho\|_{L^2}^2 + C \|\nabla_h \rho^{(0)}\|_{L^2}^2 \|(\delta v_h, \delta \rho)\|_{L^2}^2. \tag{170}
$$

Invoking the estimates in (163), (164), (165), (166), (167), (168), (169) and (170), yields

$$
\frac{d}{dt} \|(\delta v_h, \delta w, \delta \rho)\|_{L^2}^2 \leq B(t) \|(\delta v_h, \delta w, \delta \rho)\|_{L^2}^2, \tag{171}
$$

where

$$
\begin{aligned}
B(t) :=& C \|\nabla_h v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2} + C \|\partial_3 v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2} \\
& + C \|\nabla_h w\|_{L^2} \|\partial_3 \nabla_h w\|_{L^2} + C \|\partial_3 w\|_{L^2} \|\partial_3 \nabla_h w\|_{L^2} \\
& + C \|\nabla_h \rho\|_{L^2} \|\partial_3 \nabla_h \rho\|_{L^2} + C \|\nabla_h v_h\|_{L^2} \|\partial_3 \nabla_h v_h\|_{L^2}
\end{aligned}
$$

$$+ C \|\nabla_h w^{(0)}\|_{L^2}^2 + C \|\nabla_h \rho^{(0)}\|_{L^2}^2. \tag{172}$$

Gronwall's inequality then implies the desired uniqueness. This completes the proof of Proposition 2. ■

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APPENDIX: BERNSTEIN INEQUALITY, AND ODE EXISTENCE THEORY

This appendix presents several facts that have been used in the previous sections: the Bernstein inequality, existence and extension theory for ODEs on Banach spaces, and Aubin–Lions–Simon lemma. Some of these facts can be found in Ref. 5.

Lemma A.1. *Let B be a ball and C be a annulus. A constant C exists such that, for any nonnegative integer k, any $p, q \in [1, \infty]$ with $q \geq p$, and any function $f \in L^p$, we have*

$$\begin{aligned} &supp \widehat{f} \subset \lambda B \Rightarrow \|D^k f\|_{L^q} := \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \\ &supp \widehat{f} \subset \lambda C \Rightarrow C^{-k-1}\lambda^k \|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C^{k+1}\lambda^k \|f\|_{L^p}. \end{aligned} \tag{A.1}$$

Theorem A.1. *Let E be a Banach space, Ω be an open subset of E, I an open interval of $\mathbb{R}$, and $(t_0, x_0) \in I \times \Omega$. Let $F \in L^1_{loc}(I, C_\mu(\Omega, E))$, where μ is an Osgood module of continuity and $C_\mu(\Omega, E)$ is the set of bounded, continuous maps from Ω to E such that*

$$\|F(t,x)\|_{C_\mu} := \sup_{x \in \Omega} \|F(t,x)\|_E + \sup_{0<\|x-y\|_E \leq 1} \frac{\|F(t,x) - F(t,y)\|_E}{\mu(\|x-y\|_E)} < \infty. \tag{A.2}$$

Then there exists an open interval $J \subset I$ such that the ODE

$$x(t) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau \tag{A.3}$$

has a unique continuous solution.

Theorem A.2. *Let (T_*, T^*) be the maximal interval of existence, If F satisfies*

$$\|F(t,x)\|_E \leq M\, \|x\|_E^2 \tag{A.4}$$

for some constant M, then for any $t_0 \in (T_*, T^*)$, we have

$$\int_{T_*}^{t_0} \|x(t)\|_E\, dt - T_* = T^* + \int_{t_0}^{T^*} \|x(t)\|_E\, dt = \infty. \tag{A.5}$$

Lemma A.2. *Let $T > 0$. Suppose $u \in L^2(0,T;H^1(\mathbb{R}^d))$ with $\partial_t u \in L^2(0,T;H^{-1}(\mathbb{R}^d))$. Then $u \in C([0,T];L^2)$.*