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GLOBAL SOLUTIONS TO 3D INCOMPRESSIBLE MHD SYSTEM WITH DISSIPATION IN ONLY ONE DIRECTION[*]

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Abstract. The small data global well-posedness of the 3D incompressible Navier–Stokes equations in $\mathbb{R}^3$ with only one-directional dissipation remains an outstanding open problem. The dissipation in just one direction, say, $\partial_1^2 u$ is simply insufficient in controlling the nonlinearity in the whole space $\mathbb{R}^3$. The beautiful work of Paicu and Zhang [*Sci. China Math.*, 62 (2019), pp. 1175–1204] solved the case when the spatial domain is bounded in the x_1-direction by observing a crucial Poincaré-type inequality. Motivated by this Navier–Stokes open problem and by experimental observations on the stabilizing effects of background magnetic fields, this paper intends to understand the global well-posedness and stability of a special 3D magnetohydrodynamic (MHD) system near a background magnetic field. The spatial domain is $\mathbb{R}^3$, and the velocity in this MHD system obeys the 3D Navier–Stokes with only one-directional dissipation. With no Poincaré-type inequality, this problem appears to be impossible. By discovering the mathematical mechanism of the experimentally observed stabilizing effect and introducing several innovative techniques to deal with the derivative loss difficulties, we are able to bound the Navier–Stokes nonlinearity and solve the desired global well-posedness and stability problem.

Key words. 3D magnetohydrodynamic equations, mixed dissipation, global smooth solutions

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1. Introduction.

This paper focuses on a special 3D anisotropic magnetohydrodynamic (MHD) system. The velocity field obeys the 3D Navier–Stokes equation with one-directional dissipation, while the magnetic field satisfies the induction equation with two-directional magnetic diffusion. More precisely, the MHD system concerned here reads

$$\begin{cases} \partial_t u + u\cdot\nabla u - \partial_1^2 u + \nabla P = B\cdot\nabla B, \qquad x\in\mathbb{R}^3, t>0,\\ \partial_t B + u\cdot\nabla B - \partial_1^2 B - \partial_2^2 B = B\cdot\nabla u,\\ \nabla\cdot u = \nabla\cdot B = 0,\\ u(x,0)=u_0(x), \quad B(x,0)=B_0(x), \end{cases} \tag{1.1}$$

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where $u=(u_1,u_2,u_3)^T$ represents the velocity field, P the total pressure, and $B=(B_1,B_2,B_3)^T$ the magnetic field. The MHD equations reflect the basic physics laws governing the motion of electrically conducting fluids, such as plasmas, liquid metals, and electrolytes. They are a combination of the Navier–Stokes equation of fluid dynamics and Maxwell's equation of electromagnetism (see, e.g., [6, 17, 42]). The MHD system (1.1) focused on here is relevant in the modeling of reconnecting plasmas (see, e.g., [13, 14]). The Navier–Stokes equation with anisotropic viscous dissipation arises in several physical circumstances. It can model the turbulent diffusion of rotating fluids in Ekman layers. More details on the physical backgrounds of anisotropic fluids can be found in [11, 41].

The goal here is to establish the global well-posedness and stability near a background magnetic field. More precisely, the background magnetic field refers to the special steady-state solution $(u^{(0)},B^{(0)})$, where

$$u^{(0)}\equiv 0,\quad B^{(0)}\equiv (0,1,0):=e_2.$$

Any perturbation (u,b) near $(u^{(0)},B^{(0)})$ with

$$b=B-B^{(0)}$$

is governed by

$$\begin{cases}\partial_t u+u\cdot\nabla u-\partial_1^2 u+\nabla P=b\cdot\nabla b+\partial_2 b, & x\in\mathbb{R}^3, t>0,\\ \partial_t b+u\cdot\nabla b-\Delta_h b=b\cdot\nabla u+\partial_2 u, & \\ \nabla\cdot u=\nabla\cdot b=0, & \\ u(x,0)=u_0(x),\quad b(x,0)=b_0(x), & \end{cases}\tag{1.2}$$

where, for notational convenience, we have written $\Delta_h=\partial_1^2+\partial_2^2$ and we shall also write $\nabla_h=(\partial_1,\partial_2)$. Our motivation for this study comes from two sources. The first is to gain a better understanding on the well-posedness problem on the 3D Navier–Stokes equation with dissipation in only one direction. The second is to reveal and rigorously establish the stabilizing phenomenon exhibited by electrically conducting fluids. Extensive physical experiments and numerical simulations have been performed to understand the influence of the magnetic field on the bulk turbulence involving various electrically conducting fluids, such as liquid metals (see, e.g., [2, 3, 4, 5, 7, 15, 16, 17, 20, 21]). These experiments and simulations have observed a remarkable phenomenon that a background magnetic field can smooth and stabilize turbulent electrically conducting fluids. We intend to establish these observations as mathematically rigorous facts on the system (1.2).

Mathematically, the problem we are attempting appears to be impossible. The velocity satisfies the 3D incompressible forced Navier–Stokes equations with dissipation in only one direction:

$$\partial_t u+u\cdot\nabla u-\partial_1^2 u+\nabla P=b\cdot\nabla b+\partial_2 b,\quad x\in\mathbb{R}^3,\ t>0.$$

However, when the spatial domain is the whole space $\mathbb{R}^3$, the small data global well-posedness of the 3D Navier–Stokes equations with only one-directional dissipation,

$$\begin{cases}\partial_t u+u\cdot\nabla u=-\nabla p+\partial_1^2 u, & x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot u=0, & \\ u(x,0)=u_0(x), & \end{cases}\tag{1.3}$$

remains an outstanding open problem. The difficulty is immediate. The dissipation in one direction is simply not sufficient in controlling the nonlinearity when the spatial domain is the whole space $\mathbb{R}^3$.

In a beautiful work [39], Paicu and Zhang were able to deal with the case when the spatial domain is bounded in the x_1-direction with Dirichlet-type boundary conditions. They successfully established the small data global well-posedness by observing a crucial Poincaré-type inequality. This inequality allows one to bound u in terms of $\partial_1 u$ and thus leads to the control of the nonlinearity. However, such Poincaré-type inequalities are not valid for the whole space case.

If we increase the dissipation to be in two-directions, say,

$$\begin{cases}\partial_t u + u\cdot\nabla u = -\nabla p + (\partial_1^2+\partial_2^2)u, \quad x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot u = 0,\end{cases}$$

then any sufficiently small initial data in a suitable Sobolev or Besov space always leads to a global (not necessarily stable) solution. There are substantial developments on the 3D anisotropic Navier–Stokes equations with two-directional dissipation. Significant progress has been made on the global existence of small solutions and on the regularity criteria on general large solutions in various Sobolev and Besov settings (see, e.g., [11, 28, 36, 38, 39]).

We return to the well-posedness and stability problem proposed here. Clearly, if the coupling with the magnetic field did not generate an extra smoothing and stabilizing effect, then the problem focused on here would be impossible. Fortunately, we discover in this paper that the magnetic field does help stabilize the conducting fluids, as observed by physical experiments and numerical simulations. To unearth the stabilization effect, we take advantage of the coupling and intersection in the MHD system to convert (1.2) into the following wave equations:

$$\begin{cases}\partial_t^2 u - (\partial_1^2+\Delta_h)\partial_t u + \partial_1^2\Delta_h u - \partial_2^2 u = (\partial_t - \Delta_h)N_1 + \partial_2 N_2,\\ \partial_t^2 b - (\partial_1^2+\Delta_h)\partial_t b + \partial_1^2\Delta_h b - \partial_2^2 b = (\partial_t - \partial_1^2)N_2 + \partial_2 N_1,\end{cases} \tag{1.4}$$

where N_1 and N_2 are the nonlinear terms

$$N_1 = \mathbb{P}(-u\cdot\nabla u + b\cdot\nabla b), \quad N_2 = -u\cdot\nabla b + b\cdot\nabla u$$

with $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ being the Leray projection. Equation (1.4) is derived by taking the time derivative of (1.2) and making several substitutions. In comparison with (1.2), the wave structure in (1.4) exhibits many more regularity properties. In particular, the linearized wave equation of u,

$$\partial_t^2 u - (\partial_1^2+\Delta_h)\partial_t u + \partial_1^2\Delta_h u - \partial_2^2 u = 0,$$

reveals the regularization of u in the x_2-direction, although the original system (1.2) involves only the dissipation in the x_1-direction.

Unfortunately, the extra regularization in the x_2-direction is not sufficient to control the Navier–Stokes nonlinearity. The regularity from the wave structure is in general one derivative order lower than the standard dissipation. More precisely, when we seek solutions in the Sobolev space $H^4(\mathbb{R}^3)$, the dissipation in the x_1-direction yields the time integrability term

$$\int_0^t \|\partial_1 u(\tau)\|_{H^4}^2\, d\tau, \tag{1.5}$$

but the extra regularity in the x_2-direction due to the background magnetic field and the coupling can only allow us to bound

$$\int_0^t \|\partial_2 u(\tau)\|_{H^3}^2 \, d\tau. \tag{1.6}$$

But (1.5) and (1.6) may not be sufficient to control some of the terms resulting from the nonlinearity in the estimate of $\|u\|_{H^4}$, such as

$$\int_{\mathbb{R}^3} \partial_3 u_3 \, \partial_3^4 u_1 \, \partial_3^4 u_1 \, dx.$$

Naturally, we use the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ to eliminate the bad derivative ∂_3, but this process generates a new difficult term

$$\int_{\mathbb{R}^3} \partial_2 u_2 \, \partial_3^4 u_1 \, \partial_3^4 u_1 \, dx,$$

which cannot be bounded in terms of (1.6). If we integrate by parts, we would have the fifth-order derivatives on the velocity, which cannot be controlled. We call this phenomenon the derivative loss problem. The above analysis reveals that the extra regularity gained through the background magnetic field and the nonlinear coupling is not sufficient to deal with the derivative loss problem.

This paper creates several new techniques to combat the derivative loss problem. As a consequence, we are able to offer suitable upper bounds on the Navier–Stokes nonlinearity and solve the desired global well-posedness and stability problem. We state our main result and then describe these techniques.

THEOREM 1.1. *Consider* (1.2) *with the initial datum* $(u_0, b_0) \in H^4(\mathbb{R}^3)$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$*, Then there exists a constant* $\epsilon > 0$ *such that, if*

$$\|u_0\|_{H^4} + \|b_0\|_{H^4} \le \epsilon,$$

system (1.2) *has a unique global classical solution* (u, b) *satisfying, for any* $t > 0$,

$$\|u(t)\|_{H^4}^2 + \|b(t)\|_{H^4}^2 + \int_0^t \left(\|\partial_1 u\|_{H^4}^2 + \|\nabla_h b\|_{H^4}^2 + \|\partial_2 u\|_{H^3}^2 \right) d\tau \le \epsilon.$$

We make two remarks. Theorem 1.1 does not solve the small data global well-posedness problem on the 3D Navier–Stokes equations in (1.3). The original MHD system (1.1) with $B = 0$ indeed reduces to the 3D Navier–Stokes equations in (1.3). However, (1.2) governs the perturbation (u, b) and contains two extra terms $\partial_2 b$ and $\partial_2 u$. Taking $b = 0$ in (1.2) yields (1.3) together with $\partial_2 u = 0$ (resulting from the equation of b). Therefore, (1.2) with $b = 0$ reduces to the 2D Navier–Stokes equations depending on the spatial variables x_1 and x_3. It is hoped that the new idea and techniques discovered in this paper will shed light on the open problem on (1.3).

A previous work of Wu and Zhu [51] successfully resolved the small data global well-posedness and stability problem on the 3D MHD system with horizontal dissipation and vertical magnetic diffusion near an equilibrium:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta_h u + \nabla P = b \cdot \nabla b + \partial_1 b, \quad x \in \mathbb{R}^3, t > 0, \\ \partial_t b + u \cdot \nabla b - \partial_3^2 b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \tag{1.7}$$

Although the small data global well-posedness problem on (1.7) is highly nontrivial, it is clear that the current problem on (1.2) is different and can be even more difficult. One simple reason is that the equation of u in (1.7) contains dissipation in two directions and all the nonlinear terms in the estimate of the Sobolev norms involve u as a component. However, (1.2) has only one-direction velocity dissipation, and the velocity nonlinear term involves only u (no other more regularized components). The methods in [51] are not sufficient for the problem of this paper.

Since the pioneering work of Lin and Zhang [35] and Lin, Xu and Zhang [34], many efforts have now been devoted to the small data global well-posedness and stability problems on partially dissipated MHD systems. MHD systems with various levels of dissipation and magnetic diffusion near several steady states have been thoroughly investigated, and a rich array of results have been established (see, e.g., [1, 5, 8, 25, 26, 29, 30, 34, 35, 40, 43, 44, 45, 47, 49, 50, 55, 56, 57]). In addition, global well-posedness on the MHD equations with general large initial data has also been actively pursued, and important progress has been made (see, e.g., [10, 12, 18, 19, 22, 23, 24, 27, 31, 32, 33, 48, 52, 53, 54]). Needless to say, the references listed here represent only a small portion of the very large literature on the global well-posedness and related problems concerning the MHD equations.

We briefly outline the proof of Theorem 1.1. We have chosen the Sobolev space H^4 as the functional setting for our solutions since H^3 does not appear to be regular enough to accommodate our approach. Since the local well-posedness of (1.2) in H^4 follows from a standard procedure (see, e.g., [37]), the proof focuses on the global H^4-bound. The framework of the proof for the global H^4-bound is the bootstrapping argument (see, e.g., [46, page 20]). The process starts with the setup of a suitable energy functional. Besides the standard H^4-energy, we also need to include the extra regularization term in (1.6) resulting from the background magnetic field and the coupling. More precisely, we set

$$\mathscr{E}(t) = \mathscr{E}_1(t) + \mathscr{E}_2(t), \tag{1.8}$$

where

$$\begin{aligned} \mathscr{E}_1(t) &= \sup_{0\le\tau\le t} \left(\|u\|_{H^4}^2 + \|b\|_{H^4}^2\right) + \int_0^t \Big(\|\partial_1 u\|_{H^4}^2 + \|\nabla_h b\|_{H^4}^2\Big) d\tau, \\ \mathscr{E}_2(t) &= \int_0^t \|\partial_2 u\|_{H^3}^2 d\tau. \end{aligned}$$

Our main efforts are devoted to showing that, for some constant $C_0 > 0$ and for all $t > 0$,

$$\mathscr{E}(t) \le C_0\, \mathscr{E}(0) + C_0\, \mathscr{E}^{\frac{3}{2}}(0) + C_0\, \mathscr{E}^{\frac{3}{2}}(t) + C_0\, \mathscr{E}^2(t). \tag{1.9}$$

Then an application of the bootstrapping argument to (1.9) would yield the desired result; namely, for a sufficiently small $\epsilon > 0$,

$$\|u_0\|_{H^4} + \|b_0\|_{H^4} \le \epsilon \quad \text{or} \quad \mathscr{E}(0) \le \epsilon^2$$

would imply, for a constant $C > 0$ and for any $t > 0$,

$$\mathscr{E}(t) \le C\, \epsilon^2.$$

It then yields the global uniform H^4-bound on (u, b) and the stability.

The proof of (1.9) is highly nontrivial. It is achieved by establishing two inequalities for positive constants C_1 through C_8,

$$\mathscr{E}_2(t) \le C_1\,\mathscr{E}(0) + C_2\,\mathscr{E}_1(t) + C_3\,\mathscr{E}_1^{\frac{3}{2}}(t) + C_4\,\mathscr{E}_2^{\frac{3}{2}}(t), \tag{1.10}$$

$$\mathscr{E}_1(t) \le C_5\,\mathscr{E}(0) + C_6\,\mathscr{E}^{\frac{3}{2}}(0) + C_7\,\mathscr{E}^{\frac{3}{2}}(t) + C_8\,\mathscr{E}^2(t), \tag{1.11}$$

which clearly lead to (1.9) by adding (1.11) to a suitable multiple of (1.10). Due to the equivalence of the norms

$$\|f\|_{H^k} \sim \|f\|_{L^2} + \sum_{i=1}^{3} \|\partial_i^k f\|_{L^2},$$

the verification of (1.10) is naturally divided into the estimates of

$$\int_0^t \|\partial_2 u\|_{L^2}^2\, d\tau \quad \text{and} \quad \sum_{i=1}^{3} \int_0^t \|\partial_i^3 \partial_2 u\|_{L^2}^2\, d\tau.$$

One key strategy is to take advantage of the coupling and interaction of the MHD system in (1.2) to shift the time integrability. More precisely, we replace $\partial_2 u$ by the evolution of b,

$$\partial_2 u = \partial_t b + u\cdot\nabla b - \Delta_h b - b\cdot\nabla u,$$

and convert the time integral of $\|\partial_2 u\|_{L^2}^2$ into time integrals of more regular terms:

$$\int_0^t \|\partial_2 u\|_{L^2}^2\, d\tau = \int_0^t \int_{\mathbb{R}^3} (\partial_t b + u\cdot\nabla b - \Delta_h b - b\cdot\nabla u)\cdot \partial_2 u\, dx\, d\tau.$$

We then further shift the time derivative from b to $\partial_2 u$ and involve the equation of u. This process generates many more terms, but it replaces those with less time-integrable terms by more integrable nonlinear terms. More details can be found in section 2.

It is much more difficult to verify (1.11). Undoubtedly, the most difficult term is generated by the Navier–Stokes nonlinear term. We use the vorticity formulation to take advantage of certain symmetries. Since $\|\omega\|_{L^2} = \|\nabla u\|_{L^2}$ for the vorticity $\omega = \nabla\times u$, it suffices to control $\|\omega\|_{\dot H^3}$. One of the wildest terms is given by

$$\int \partial_3 u_3\, \partial_3^3\omega\cdot\partial_3^3\omega\, dx.$$

Naturally, we eliminate the bad derivative ∂_3 via the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, but this leads to another uncontrolled term:

$$\int \partial_2 u_2\, \partial_3^3\omega\cdot\partial_3^3\omega\, dx. \tag{1.12}$$

As we have commented before, integrating by parts in (1.12) would generate fourth-order derivatives of the vorticity (or fifth-order derivatives of the velocity) such as $\partial_2\partial_3^3\omega$, which cannot be controlled by $\mathscr{E}_2$. Obtaining a suitable bound on (1.12) appears to be an impossible mission. This leads to the derivative loss problem. The main thrust of this paper is to introduce several new techniques to unearth the hidden structure in the nonlinearity. We enhance the nonlinearity through the coupling in the system and induce cancellations through the construction of artificial symmetries. More precisely, we are able to establish the desired upper bounds stated in the following proposition. For notational convenience, we use $A \lesssim B$ to mean $A \le C\,B$ for a pure constant $C > 0$.

PROPOSITION 1.1. *Let $(u,b)\in H^4$ be a solution of (1.2). Let $\omega=\nabla\times u$ and $H=\nabla\times b$ be the corresponding vorticity and current density, respectively. Let $\mathscr{E}(t)$ be defined as in (1.8). Let $\mathcal{W}(t)$ be the interaction type terms defined as follows:*

$$\mathcal{W}^{ijk}(t)=\int_{\mathbb{R}^3}\partial_3^3\omega_i\partial_2 u_j\partial_3^3\omega_k\,dx\quad \text{for } i,j,k\in\{1,2,3\}.$$

Then the time integral of $\mathcal{W}^{ijk}$ admits the following bound:

$$\int_0^t \mathcal{W}^{ijk}(\tau)\,d\tau\lesssim \mathscr{E}^{\frac{3}{2}}(0)+\mathscr{E}^{\frac{3}{2}}(t)+\mathscr{E}^2(t).$$

The proof of this proposition is not trivial. When we just have the 3D Navier–Stokes with one-directional dissipation, this term cannot be suitably bounded, and the small-data global well-posedness remains an open problem for the 3D Navier–Stokes. The advantage of working with the MHD system in (1.2) is the coupling and interaction. We take advantage of this coupling to replace $\partial_2 u_j$ via the equation of b:

$$\begin{aligned}\mathcal{W}^{ijk}(t)&=\int_{\mathbb{R}^3}\partial_3^3\omega_i\Big[\partial_t b_j+u\cdot\nabla b_j-\Delta_h b_j-b\cdot\nabla u_j\Big]\partial_3^3\omega_k\,dx\\&=\frac{d}{dt}\int_{\mathbb{R}^3}\partial_3^3\omega_i b_j\partial_3^3\omega_k dx-\int_{\mathbb{R}^3}b_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)dx\\&\quad+\int_{\mathbb{R}^3}\partial_3^3\omega_i u\cdot\nabla b_j\partial_3^3\omega_k dx-\int_{\mathbb{R}^3}\partial_3^3\omega_i b\cdot\nabla u_j\partial_3^3\omega_k dx\\&\quad-\int_{\mathbb{R}^3}\partial_3^3\omega_i\Delta_h b_j\partial_3^3\omega_k dx.\end{aligned}$$

Immediately, we encounter the new difficult term

$$-\int_{\mathbb{R}^3}b_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)dx,$$

which is further converted into integrals of many more terms after invoking the evolution of the vorticity:

$$\begin{aligned}&-\int_{\mathbb{R}^3}b_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)dx\\&=\int_{\mathbb{R}^3}b_j\partial_3^3\omega_i\partial_3^3\big(u\cdot\nabla\omega_k-\omega\cdot\nabla u_k-\partial_1^2\omega_k-b\cdot\nabla H_k+H\cdot\nabla b_k-\partial_2 H_k\big)\\&\quad+b_j\,\partial_3^3\omega_k\partial_3^3\big(u\cdot\nabla\omega_i-\omega\cdot\nabla u_i-\partial_1^2\omega_i-b\cdot\nabla H_i+H\cdot\nabla b_i-\partial_2 H_i\big)\,dx.\end{aligned}$$

Some of the terms above can be paired together to form symmetries to deal with the derivative loss problem. This process also allows us to convert some of the cubic nonlinearity into quartic nonlinearity, which helps improve the time integrability. However, there are some terms that cannot be paired into symmetric structure, for example,

$$\int_{\mathbb{R}^3}b_j\left[-\partial_3^3\omega_i b\cdot\nabla\partial_3^3 H_k-\partial_3^3\omega_k b\cdot\nabla\partial_3^3 H_i\right]\,dx.$$

Our idea in dealing with such terms is to construct artificial symmetries by adding and subtracting suitable terms. This strategy helps us alleviate the derivative loss problem eventually. The technical details are very complicated and are left to the proof of Proposition 1.1 in section 4.

The rest of this paper is divided into four sections. Section 2 proves (1.10), one of the two key energy inequalities, while section 3 establishes the second key energy inequality, namely, (1.11). Section 4 provides the proof of Proposition 1.1 and deals with some of the most difficult terms in the Navier–Stokes nonlinearity in Lemma 4.1. The last section, section 5, finishes the proof of Theorem 1.1.

2. Proof of (1.10).

This section proves (1.10); namely, for four positive constants C_1 through C_4,

$$\mathscr{E}_2 \le C_1\,\mathscr{E}(0) + C_2\,\mathscr{E}_1(t) + C_3\,\mathscr{E}_1^{\frac{3}{2}}(t) + C_4\,\mathscr{E}_2^{\frac{3}{2}}(t) \quad \text{for all } t>0. \tag{2.1}$$

The following lemma provides a powerful tool to control the triple products in terms of anisotropic upper bounds.

Lemma 2.1. *The following inequalities hold when the right-hand sides are all bounded:*

$$\begin{aligned}
\int_{\mathbb{R}^3} |fgh|\,dx &\lesssim \|f\|_{L^2}^{\frac{1}{2}}\|\partial_1 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_3 h\|_{L^2}^{\frac{1}{2}},\\
\int_{\mathbb{R}^3} |fgh|\,dx &\lesssim \|f\|_{L^2}^{\frac{1}{4}}\|\partial_i f\|_{L^2}^{\frac{1}{4}}\|\partial_j f\|_{L^2}^{\frac{1}{4}}\|\partial_i\partial_j f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_k g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}\\
&\lesssim \|f\|_{H^1}^{\frac{1}{2}}\|\partial_i f\|_{H^1}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_k g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2},\\
\left(\int_{\mathbb{R}^3} |fg|^2\,dx\right)^{\frac{1}{2}} &\lesssim \|f\|_{L^2}^{\frac{1}{4}}\|\partial_i f\|_{L^2}^{\frac{1}{4}}\|\partial_j f\|_{L^2}^{\frac{1}{4}}\|\partial_i\partial_j f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_k g\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|f\|_{H^1}^{\frac{1}{2}}\|\partial_i f\|_{H^1}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_k g\|_{L^2}^{\frac{1}{2}},\\
\int_{\mathbb{R}^3} |fghv|\,dx &\lesssim \|f\|_{L^2}^{\frac{1}{4}}\|\partial_i f\|_{L^2}^{\frac{1}{4}}\|\partial_j f\|_{L^2}^{\frac{1}{4}}\|\partial_i\partial_j f\|_{L^2}^{\frac{1}{4}}\\
&\qquad\cdot \|g\|_{L^2}^{\frac{1}{4}}\|\partial_i g\|_{L^2}^{\frac{1}{4}}\|\partial_j g\|_{L^2}^{\frac{1}{4}}\|\partial_i\partial_i g\|_{L^2}^{\frac{1}{4}}\\
&\qquad\cdot \|h\|_{L^2}^{\frac{1}{2}}\|\partial_k h\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\partial_k v\|_{L^2}^{\frac{1}{2}}\\
&\lesssim \|f\|_{H^1}^{\frac{1}{2}}\|\partial_i f\|_{H^1}^{\frac{1}{2}}\|g\|_{H^1}^{\frac{1}{2}}\|\partial_i g\|_{H^1}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_k h\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\partial_k v\|_{L^2}^{\frac{1}{2}},
\end{aligned}$$

where i, j, and k, belonging to $\{1,2,3\}$, are different numbers.

The proof of Lemma 2.1 relies on the following 1D interpolation inequality for $f \in H^1(\mathbb{R})$:

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

A detailed proof of this lemma can be found in [51]. We remark that similar anisotropic inequalities are also available for 2D functions (see [9]).

We are now ready to prove (2.1).

Proof of (2.1). Due to the norm equivalence, namely, for any integer $k>0$,

$$\|f\|_{H^k}^2 \sim \|f\|_{L^2}^2 + \sum_{i=1}^{3}\|\partial_i^k f\|_{L^2}^2, \tag{2.2}$$

it suffices to bound

$$\int_0^t \|\partial_2 u\|_{L^2}^2\,d\tau \quad \text{and} \quad \sum_{i=1}^{3}\int_0^t \|\partial_i^3\partial_2 u\|_{L^2}^2\,d\tau.$$

Recalling the equations in (1.2),

$$\begin{aligned} \partial_2 u &= \partial_t b + u\cdot\nabla b - \Delta_h b - b\cdot\nabla u, \\ \partial_t u &= -u\cdot\nabla u + \partial_1^2 u - \nabla P + b\cdot\nabla b + \partial_2 b, \end{aligned} \tag{2.3}$$

we have

$$\begin{aligned} \|\partial_2 u\|_{L^2}^2 &= \int_{\mathbb{R}^3} (\partial_t b + u\cdot\nabla b - \Delta_h b - b\cdot\nabla u)\cdot\partial_2 u\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}^3} b\cdot\partial_2 u\,dx + \int_{\mathbb{R}^3} \partial_2 b\cdot\partial_t u dx \\ &\quad + \int_{\mathbb{R}^3} (u\cdot\nabla b - \Delta_h b - b\cdot\nabla u)\cdot\partial_2 u\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}^3} b\cdot\partial_2 u\,dx \\ &\quad + \int_{\mathbb{R}^3} \partial_2 b\cdot(\partial_2 b + b\cdot\nabla b - \nabla P + \partial_1^2 u - u\cdot\nabla u)\,dx \\ &\quad + \int_{\mathbb{R}^3} (u\cdot\nabla b - \Delta_h b - b\cdot\nabla u)\cdot\partial_2 u\,dx. \end{aligned} \tag{2.4}$$

Due to $\nabla\cdot b = 0$,

$$\int_{\mathbb{R}^3} \partial_2 b\cdot\nabla P\,dx = 0.$$

We bound the nonlinear terms. By Sobolev's inequality and Lemma 2.1,

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_2 b\cdot(b\cdot\nabla b)\,dx &= \int_{\mathbb{R}^3} \big(\partial_2 b\cdot(b_h\cdot\nabla_h b) + \partial_2 b\cdot(b_3\partial_3 b)\big)\,dx \\ &\lesssim \|\nabla_h b\|_{L^2}^2\|b\|_{H^2} + \|\partial_2 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2 b\|_{L^2}^{\frac12}\|b_3\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}} b_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3 b\|_{L^2}^{\frac12} \\ &\lesssim \|b\|_{H^2}\|\nabla_h b\|_{H^1}^2, \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_2 b\cdot(u\cdot\nabla u)\,dx &= \int_{\mathbb{R}^3} \big(\partial_2 b\cdot(u_h\cdot\nabla_h u) + \partial_2 b\cdot(u_3\partial_3 u)\big)\,dx \\ &\lesssim \|\partial_2 b\|_{L^2}\|\nabla_h u\|_{L^2}\|u\|_{H^2} + \|\partial_2 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2 b\|_{L^2}^{\frac12}\|u_3\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}} u_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3 u\|_{L^2}^{\frac12} \\ &\lesssim \|\partial_2 b\|_{L^2}\|\nabla_h u\|_{L^2}\|u\|_{H^2} + \|\partial_2 b\|_{H^1}\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1 u\|_{H^1}^{\frac12}\|u\|_{H^1}, \end{aligned}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} (u\cdot\nabla b)\cdot\partial_2 u\,dx \\ &\quad = \int_{\mathbb{R}^3} \big((u_h\cdot\nabla_h b)\cdot\partial_2 u + (u_3\partial_3 b)\cdot\partial_2 u\big)\,dx \\ &\quad \lesssim \|\nabla_h b\|_{L^2}\|\partial_2 u\|_{L^2}\|u\|_{H^2} + \|u_3\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}} u_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3 b\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_2\partial_{\mathbf{3}} u\|_{L^2}^{\frac12}, \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} (b\cdot\nabla u)\cdot\partial_2 u\,dx \\ &\quad = \int_{\mathbb{R}^3} (b_h\cdot\nabla_h u)\cdot\partial_2 u + (b_3\partial_3 u)\cdot\partial_2 u dx \\ &\quad \lesssim \|b\|_{H^2}\|\nabla_h u\|_{L^2}^2 + \|b_3\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}} b_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3 u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_2\partial_{\mathbf{3}} u\|_{L^2}^{\frac12}. \end{aligned}$$

Inserting these bounds in (2.4), integrating in time, and using the simple bound

$$\int_0^t \frac{d}{d\tau} \int_{\mathbb{R}^3} b \cdot \partial_2 u \, dx \, d\tau \le \|b(t)\|_{L^2} \|\partial_2 u(t)\|_{L^2} + \|b_0\|_{L^2} \|\partial_2 u_0\|_{L^2} \le \mathscr{E}_1(t) + \mathscr{E}_1(0),$$

we obtain

$$\begin{aligned} \int_0^t \|\partial_2 u\|_{L^2}^2 \, d\tau &\lesssim \mathscr{E}_1(0) + \mathscr{E}_1(t) + + \mathscr{E}_1^{\frac{3}{2}}(t) + \mathscr{E}_1(t) \mathscr{E}_2^{\frac{1}{2}}(t) \\ &\quad + \mathscr{E}_1^{\frac{1}{2}}(t) \mathscr{E}_2(t) + \mathscr{E}_1^{\frac{5}{4}}(t) \mathscr{E}_2^{\frac{1}{4}}(t) \\ &\lesssim \mathscr{E}_1(0) + \mathscr{E}_1(t) + \mathscr{E}_1^{\frac{3}{2}}(t) + \mathscr{E}_2^{\frac{3}{2}}(t). \end{aligned} \tag{2.5}$$

Here we have used several Hölder's inequalities, such as

$$\sup_{0 \le \tau \le t} \|b(\tau)\|_{H^2} \int_0^t \|\nabla_h b\|_{H^1}^2 \, d\tau \le \mathscr{E}_1^{\frac{3}{2}}(t),$$

$$\sup_{0 \le \tau \le t} \|u(\tau)\|_{H^2} \int_0^t \|\partial_2 b\|_{L^2} \|\partial_2 u\|_{L^2} \, d\tau \le \mathscr{E}_1(t) \mathscr{E}_2^{\frac{1}{2}}(t) \le \mathscr{E}_1^{\frac{3}{2}}(t) + \mathscr{E}_2^{\frac{3}{2}}(t).$$

We now turn to the bound for the highest-order derivatives. By (2.3),

$$\begin{aligned} \sum_{i=1}^3 \|\partial_i^3 \partial_2 u\|_{L^2}^2 &= \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 \Big(\partial_t b + u \cdot \nabla b - \Delta_h b - b \cdot \nabla u \Big) \cdot \partial_i^3 \partial_2 u dx \\ &= \sum_{i=1}^3 \frac{d}{dt} \int_{\mathbb{R}^3} \partial_i^3 b \cdot \partial_i^3 \partial_2 u dx + \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 \partial_2 b \cdot \partial_i^3 \partial_t u \, dx \\ &\quad + \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 \Big(u \cdot \nabla b - \Delta_h b - b \cdot \nabla u \Big) \cdot \partial_i^3 \partial_2 u dx. \end{aligned} \tag{2.6}$$

The estimates for the terms with $i = 1, 2$ (those containing ∂_1^3 or ∂_2^3 derivatives) are simple. We focus on the terms with $i = 3$ (those with the bad derivative ∂_3), namely,

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} \partial_3^3 b \cdot \partial_3^3 \partial_2 u dx + \int_{\mathbb{R}^3} \partial_3^3 \partial_2 b \cdot \partial_3^3 u_t dx \\ &+ \int_{\mathbb{R}^3} \partial_3^3 \Big(u \cdot \nabla b - \Delta_h b - b \cdot \nabla u \Big) \cdot \partial_3^3 \partial_2 u dx. \end{aligned} \tag{2.7}$$

The second part of (2.7) can be handled as follows. By Lemma 2.1,

$$\begin{aligned} &\int_{\mathbb{R}^3} \partial_3^3 \partial_2 b \cdot \partial_3^3 \partial_t u \, dx \\ &\quad = \int_{\mathbb{R}^3} \partial_3^3 \partial_2 b \cdot \partial_3^3 (\partial_2 b + b \cdot \nabla b - \nabla P + \partial_1^2 u - u \cdot \nabla u) dx \\ &\quad \lesssim \|\partial_2 b\|_{H^3} \big(\|\partial_2 b\|_{H^3} + \|\partial_1 u\|_{H^4} \big) \\ &\qquad + \|\partial_2 b\|_{H^3}^{\frac{1}{2}} \|\boldsymbol{\partial_3} \partial_2 b\|_{H^3}^{\frac{1}{2}} \big(\|b\|_{H^3}^{\frac{1}{2}} \|\boldsymbol{\partial_2} b\|_{H^3}^{\frac{1}{2}} \|b\|_{H^4}^{\frac{1}{2}} \|\boldsymbol{\partial_1} b\|_{H^4}^{\frac{1}{2}} \\ &\qquad + \|u\|_{H^3}^{\frac{1}{2}} \|\boldsymbol{\partial_2} u\|_{H^3}^{\frac{1}{2}} \|u\|_{H^4}^{\frac{1}{2}} \|\boldsymbol{\partial_1} u\|_{H^4}^{\frac{1}{2}} \big). \end{aligned}$$

The last part in (2.7) can be bounded by

$$\begin{aligned}&\int_{\mathbb{R}^3} \partial_3^3\Big(u\cdot\nabla b-\Delta_h b-b\cdot\nabla u\Big)\cdot\partial_3^3\partial_2 u dx\\ &\quad\lesssim \|\partial_2 u\|_{H^3}\big(\|\nabla_h b\|_{H^4}+\|u\|_{H^3}^{\frac14}\|\partial_{\mathbf{1}} u\|_{H^3}^{\frac14}\|\partial_{\mathbf{3}} u\|_{H^3}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{3}} u\|_{H^3}^{\frac14}\|b\|_{H^4}^{\frac12}\|\partial_{\mathbf{2}} b\|_{H^4}^{\frac12}\\ &\qquad+\|b\|_{H^3}^{\frac14}\|\partial_{\mathbf{2}} b\|_{H^3}^{\frac14}\|\partial_{\mathbf{3}} b\|_{H^3}^{\frac14}\|\partial_{\mathbf{2}}\partial_{\mathbf{3}} b\|_{H^3}^{\frac14}\|u\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}} u\|_{H^4}^{\frac12}\big).\end{aligned}$$

The terms with $i=1,2$ in (2.6) are simpler and can be bounded similarly by applying Lemma 2.1. Inserting the bounds above in (2.6) and integrating in time yields

$$\sum_{i=1}^{3}\int_0^t \|\partial_i^3\partial_2 u\|_{L^2}^2 d\tau \lesssim \mathscr{E}_1(0)+\mathscr{E}_1(t)+\mathscr{E}_1^{\frac32}(t)+\mathscr{E}_2^{\frac32}(t). \tag{2.8}$$

Combining (2.5) and (2.8) gives (2.1). This finishes the proof for (1.10). ☐

3. Proof of (1.11).

This section proves the energy inequality in (1.11).

Proof of (1.11). Due to the norm equivalence (2.2), it suffices to bound

$$\sup_{0\le\tau\le t}\big(\|u\|_{L^2}^2+\|b\|_{L^2}^2\big)+\int_0^t\Big(\|\partial_1 u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2\Big)d\tau$$

and

$$\sup_{0\le\tau\le t}\sum_{i=1}^{3}\big(\|\partial_i^3\omega\|_{L^2}^2+\|\partial_i^3 H\|_{L^2}^2\big)+\sum_{i=1}^{3}\int_0^t\Big(\|\partial_i^3\partial_1\omega\|_{L^2}^2+\|\partial_i^3\nabla_h H\|_{L^2}^2\Big)d\tau,$$

where $\omega=\nabla\times u$ and $H=\nabla\times b$ are the vorticity and the current density, respectively. As aforementioned, $\|\omega\|_{L^2}=\|\nabla u\|_{L^2}$ and $\|H\|_{L^2}=\|\nabla b\|_{L^2}$.

Taking the inner product of (u,b) with the first two equations of (1.2), integrating by parts and using $\nabla\cdot u=\nabla\cdot b=0$, and then integrating in time, we find

$$\|u\|_{L^2}^2+\|b\|_{L^2}^2+2\int_0^t\Big(\|\partial_1 u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2\Big)d\tau=\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2. \tag{3.1}$$

Applying the operator $\nabla\times$ to (1.2), we obtain the system governing (ω,H):

$$\begin{cases}\partial_t\omega+u\cdot\nabla\omega-\omega\cdot\nabla u-\partial_1^2\omega=b\cdot\nabla H-H\cdot\nabla b+\partial_2 H,\\ \partial_t H+\nabla\times(u\cdot\nabla b)-\Delta_h H=\nabla\times(b\cdot\nabla u)+\partial_2\omega.\end{cases} \tag{3.2}$$

Applying ∂_i^3 with $i=1,2,3$ to (3.2) and taking the inner product of $(\partial_i^3\omega,\partial_i^3 H)$ with the resulting equations, we have, after integration by parts,

$$\begin{aligned}&\frac12\frac{d}{dt}\sum_{i=1}^{3}\Big[\|\partial_i^3\omega\|_{L^2}^2+\|\partial_i^3 H\|_{L^2}^2\Big]+\sum_{i=1}^{3}\Big[\|\partial_i^3\partial_1\omega\|_{L^2}^2+\|\partial_i^3\nabla_h H\|_{L^2}^2\Big]\\ &\quad=\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3\big[-u\cdot\nabla\omega+\omega\cdot\nabla u+b\cdot\nabla H-H\cdot\nabla b+\partial_2 H\big]\cdot\partial_i^3\omega dx\\ &\qquad+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3\big[-\nabla\times(u\cdot\nabla b)+\nabla\times(b\cdot\nabla u)+\partial_2\omega\big]\cdot\partial_i^3 H dx\\ &\quad=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.\end{aligned} \tag{3.3}$$

For the first term I_1, it can be written into three parts:

$$\int_{\mathbb{R}^3} \partial_1^3(-u\cdot\nabla\omega)\cdot\partial_1^3\omega dx,\ \int_{\mathbb{R}^3} \partial_2^3(-u\cdot\nabla\omega)\cdot\partial_2^3\omega dx,\ \int_{\mathbb{R}^3} \partial_3^3(-u\cdot\nabla\omega)\cdot\partial_3^3\omega dx.$$

The first and second parts above behave well and can be bounded easily:

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3} \partial_1^3(-u\cdot\nabla\omega)\cdot\partial_1^3\omega dx\, d\tau + \int_0^t\int_{\mathbb{R}^3} \partial_2^3(-u\cdot\nabla\omega)\cdot\partial_2^3\omega dx\, d\tau\\ &\lesssim \sup_{0\le\tau\le t}\|u(\tau)\|_{H^4}\int_0^t \|\nabla_h\omega\|_{\dot H^2}^2\, d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(t).\end{aligned}$$

We then turn to the hard term $\int_{\mathbb{R}^3}\partial_3^3(-u\cdot\nabla\omega)\cdot\partial_3^3\omega dx$ in I_1. It is difficult to control since there is no dissipation in the x_3 direction. We further decompose it into three parts:

$$\begin{aligned}&\int_{\mathbb{R}^3} \partial_3^3(-u\cdot\nabla\omega)\cdot\partial_3^3\omega dx\\ &\quad = \sum_{k=1}^3 \int_{\mathbb{R}^3} -\mathcal{C}_3^k\partial_3^k u\cdot\nabla\partial_3^{3-k}\omega\cdot\partial_3^3\omega dx\\ &\quad = -\left\{\sum_{k=2}^3 \mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k u_h\cdot\nabla_h\partial_3^{3-k}\omega\cdot\partial_3^3\omega dx + 3\int_{\mathbb{R}^3}\partial_3 u_h\cdot\nabla_h\partial_3^2\omega\cdot\partial_3^3\omega dx\right\}\\ &\quad -\sum_{k=2}^3\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k u_3\partial_3^{4-k}\omega\cdot\partial_3^3\omega dx - 3\int_{\mathbb{R}^3}\partial_3 u_3\partial_3^3\omega\cdot\partial_3^3\omega dx\\ &\quad = I_{11}+I_{12}+I_{13}.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}|I_{11}| &\lesssim \sum_{k=2}^3 \|\partial_3^k u_h\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^k u_h\|_{L^2}^{\frac12}\|\nabla_h\partial_3^{3-k}\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\nabla_h\partial_3^{3-k}\omega\|_{L^2}^{\frac12}\|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}\\ &\quad + \|\partial_3 u_h\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}}\partial_3 u_h\|_{L^2}^{\frac14}\|\partial_{\mathbf{3}}\partial_3 u_h\|_{L^2}^{\frac14}\|\partial_{\mathbf{2}}\partial_{\mathbf{3}}\partial_3 u_h\|_{L^2}^{\frac14}\|\nabla_h\partial_3^2\omega\|_{L^2}\|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}.\end{aligned}$$

Integrating in time and applying Hölder's inequality yields

$$\int_0^t |I_{11}|d\tau \lesssim \mathscr{E}^{\frac32}(t).$$

Similarly,

$$\begin{aligned}I_{12} &= \sum_{k=2}^3\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^{k-1}\nabla_h\cdot u_h\partial_3^{4-k}\omega\cdot\partial_3^3\omega dx\\ &\lesssim \sum_{k=2}^3 \|\partial_3^{k-1}\nabla_h\cdot u_h\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_3^{k-1}\nabla_h\cdot u_h\|_{L^2}^{\frac12}\\ &\quad\cdot\|\partial_3^{4-k}\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^{4-k}\omega\|_{L^2}^{\frac12}\|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}.\end{aligned}$$

Therefore,

$$\int_0^t |I_{12}|d\tau \lesssim \mathscr{E}^{\frac32}(t).$$

By the divergence-free condition $\nabla \cdot u = 0$, we can further split I_{13} into two parts:

$$I_{13} = -\int_{\mathbb{R}^3} \partial_1 u_1 \partial_3^3 \omega \cdot \partial_3^3 \omega dx - \int_{\mathbb{R}^3} \partial_2 u_2 \partial_3^3 \omega \cdot \partial_3^3 \omega dx = I_{131} + I_{132}.$$

Using integration by parts and Lemma 2.1, we have

$$\begin{aligned}\int_0^t I_{131}\, d\tau &= 2\int_0^t \int_{\mathbb{R}^3} u_1 \partial_3^3 \omega \cdot \partial_1 \partial_3^3 \omega dx \\ &\lesssim \int_0^t \|u_1\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_2} u_1\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_3} u_1\|_{L^2}^{\frac{1}{4}} \|\boldsymbol{\partial_2 \partial_3} u_1\|_{L^2}^{\frac{1}{4}} \\ &\qquad \cdot \|\partial_3^3 \omega\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{\partial_1} \partial_3^3 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3^3 \omega\|_{L^2}\, d\tau \\ &\lesssim \mathscr{E}^{\frac{3}{2}}(t).\end{aligned}$$

However, I_{132} cannot be similarly treated as I_{131}. Integration by parts would generate $\partial_2 \partial_3^3 \omega$, which cannot be bounded by either $\mathscr{E}_1$ or $\mathscr{E}_2$. We call this trouble the derivative loss problem. How to overcome the derivative loss problem is the main challenge of our proof. By creating several new techniques, we are able to deal with this type of terms. This is done in Proposition 1.1. Therefore, by Proposition 1.1, we have

$$\int_0^t I_{132}\, d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(0) + \mathscr{E}^{\frac{3}{2}}(t) + \mathscr{E}^2(t).$$

Collecting all the upper bounds for various parts of I_1, we obtain

$$\int_0^t |I_1(\tau)| d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(0) + \mathscr{E}^{\frac{3}{2}}(t) + \mathscr{E}^2(t). \tag{3.4}$$

We turn to the second term I_2. Naturally, we divide it into the following three parts:

$$\int_{\mathbb{R}^3} \partial_1^3 (\omega \cdot \nabla u) \cdot \partial_1^3 \omega dx, \quad \int_{\mathbb{R}^3} \partial_2^3 (\omega \cdot \nabla u) \cdot \partial_2^3 \omega dx, \quad \text{and} \quad \int_{\mathbb{R}^3} \partial_3^3 (\omega \cdot \nabla u) \cdot \partial_3^3 \omega dx.$$

The first two parts above can be controlled easily like before:

$$\int_0^t \int_{\mathbb{R}^3} \nabla_h^3 (\omega \cdot \nabla u) \cdot \nabla_h^3 \omega\, dx dt \lesssim \int_0^t \|u\|_{H^3} \|\nabla_h u\|_{H^3} \|\nabla_h^3 \omega\|_{L^2}\, d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(t).$$

The last part is further split into three terms as follows:

$$\begin{aligned}\int_{\mathbb{R}^3} \partial_3^3 (\omega \cdot \nabla u) \cdot \partial_3^3 \omega dx &= \int_{\mathbb{R}^3} \partial_3^3 (\omega_1 \partial_1 u) \cdot \partial_3^3 \omega dx + \int_{\mathbb{R}^3} \partial_3^3 (\omega_2 \partial_2 u) \cdot \partial_3^3 \omega dx \\ &\quad + \int_{\mathbb{R}^3} \partial_3^3 (\omega_3 \partial_3 u) \cdot \partial_3^3 \omega dx \\ &= I_{21} + I_{22} + I_{23}.\end{aligned} \tag{3.5}$$

The estimate for I_{21} is not difficult. By integration by parts,

$$\begin{aligned}\int_0^t I_{21}(\tau)\,d\tau &= \sum_{k=0}^{2} \mathcal{C}_3^k \int_0^t \int_{\mathbb{R}^3} \partial_3^k \omega_1 \partial_1 \partial_3^{3-k} u \cdot \partial_3^3 \omega \, dx d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \partial_3^3 \omega_1 \partial_1 u \cdot \partial_3^3 \omega \, dx d\tau \\ &\lesssim \sum_{k=0}^{2} \int_0^t \|\partial_3^k \omega_1\|_{L^2}^{\frac12} \|\partial_{\mathbf{2}} \partial_3^k \omega_1\|_{L^2}^{\frac12} \|\partial_1 \partial_3^{3-k} u\|_{L^2}^{\frac12} \|\partial_{\mathbf{3}} \partial_1 \partial_3^{3-k} u\|_{L^2}^{\frac12} \\ &\qquad \cdot \|\partial_3^3 \omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}} \partial_3^3 \omega\|_{L^2}^{\frac12} \, d\tau \\ &\qquad + \int_0^t \|u\|_{L^2}^{\frac14} \|\partial_{\mathbf{2}} u\|_{L^2}^{\frac14} \|\partial_{\mathbf{3}} u\|_{L^2}^{\frac14} \|\partial_{\mathbf{2}} \partial_{\mathbf{3}} u\|_{L^2}^{\frac14} \|\partial_1 \partial_3^3 \omega\|_{L^2} \\ &\qquad\quad \cdot \|\partial_3^3 \omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}} \partial_3^3 \omega\|_{L^2}^{\frac12} \, d\tau \\ &\lesssim \mathscr{E}^{\frac32}(t).\end{aligned}$$

I_{22} contains a difficult term that has to be dealt with by Proposition 1.1. By Lemma 2.1 and Proposition 1.1,

$$\begin{aligned}&\int_0^t I_{22}(\tau)\,d\tau \\ &\quad = \sum_{k=0}^{2} \mathcal{C}_3^k \int_0^t \int_{\mathbb{R}^3} \partial_3^k \omega_2 \partial_2 \partial_3^{3-k} u \cdot \partial_3^3 \omega \, dx d\tau + \int_0^t \int_{\mathbb{R}^3} \partial_3^3 \omega_2 \partial_2 u \cdot \partial_3^3 \omega \, dx d\tau \\ &\quad \lesssim \int_0^t \|\omega_2\|_{H^2}^{\frac12} \|\partial_{\mathbf{2}} \omega\|_{H^2}^{\frac12} \|\partial_2 u\|_{H^3} \|\partial_3^3 \omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}} \partial_3^3 \omega\|_{L^2}^{\frac12} \, d\tau + \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t) \\ &\quad \lesssim \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t).\end{aligned}$$

Now we come to deal with the last part in (3.5), i.e.,

$$I_{23} = \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k \omega_3 \partial_3^{4-k} u \cdot \partial_3^3 \omega \, dx + \int_{\mathbb{R}^3} \omega_3 \partial_3^4 u \cdot \partial_3^3 \omega \, dx = I_{231} + I_{232}.$$

Due to the divergence-free condition of ω, I_{231} is easy to control. By $\nabla \cdot \omega = 0$,

$$\begin{aligned}I_{231} &= -\sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^{k-1} (\partial_1 \omega_1 + \partial_2 \omega_2) \partial_3^{4-k} u \cdot \partial_3^3 \omega \, dx \\ &\quad - \int_{\mathbb{R}^3} \partial_3^2 (\partial_1 \omega_1 + \partial_2 \omega_2) \partial_3 u \cdot \partial_3^3 \omega \, dx \\ &\lesssim \|\nabla_h \omega\|_{H^1}^{\frac12} \|\partial_{\mathbf{3}} \nabla_h \omega\|_{H^1}^{\frac12} \|u\|_{H^3}^{\frac12} \|\partial_{\mathbf{2}} u\|_{H^3}^{\frac12} \|\partial_3^3 \omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}} \partial_3^3 \omega\|_{L^2}^{\frac12} \\ &\quad + \|\partial_3^2 \nabla_h \omega\|_{L^2} \|\partial_3 u\|_{L^2}^{\frac14} \|\partial_{\mathbf{2}} \partial_3 u\|_{L^2}^{\frac14} \|\partial_{\mathbf{3}} \partial_3 u\|_{L^2}^{\frac14} \|\partial_{\mathbf{2}} \partial_{\mathbf{3}} \partial_3 u\|_{L^2}^{\frac14} \|\partial_3^3 \omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}} \partial_3^3 \omega\|_{L^2}^{\frac12}.\end{aligned}$$

I_{232} cannot be bounded in the similar way. We write $\omega_3 = \partial_1 u_2 - \partial_2 u_1$ in I_{232}:

$$\int_0^t I_{232}(\tau)\,d\tau = \int_0^t \int_{\mathbb{R}^3} (\partial_1 u_2 - \partial_2 u_1) \partial_3^4 u \cdot \partial_3^3 \omega dx d\tau.$$

The estimate for $\int_0^t \int_{\mathbb{R}^3} \partial_1 u_2 \partial_3^4 u \cdot \partial_3^3 \omega dx d\tau$ is just similar to $\int_0^t \int_{\mathbb{R}^3} \partial_3^3 \omega_1 \partial_1 u \cdot \partial_3^3 \omega dx d\tau$ in I_{21}. We can apply integration by parts and Lemma 2.1 to control it by $\mathscr{E}^{\frac32}(t)$.

It remains difficult to deal with $\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u\cdot\partial_3^3\omega dxd\tau$ due to the derivative loss problem. We split this term into three parts again:

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u\cdot\partial_3^3\omega dxd\tau =&\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u_1\partial_3^3\omega_1 dxd\tau\\
&+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u_2\partial_3^3\omega_2 dxd\tau\\
&+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u_3\partial_3^3\omega_3 dxd\tau.
\end{aligned}\tag{3.6}
$$

The last part $\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u_3\partial_3^3\omega_3 dxd\tau$ can be controlled by $\mathscr{E}^{\frac{3}{2}}(t)$ easily by making use of the divergence-free property of u. The process is simpler than that for I_{231}. Now we focus on the first two parts in (3.6). We write them as

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u_1\partial_3^3\omega_1 dxd\tau+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^4 u_2\partial_3^3\omega_2 dxd\tau\\
&\quad=\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\big(\partial_3 u_1-\partial_1 u_3+\partial_1 u_3\big)\partial_3^3\omega_1 dxd\tau\\
&\qquad+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\big(\partial_3 u_2-\partial_2 u_3+\partial_2 u_3\big)\partial_3^3\omega_2 dxd\tau\\
&\quad=\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\omega_2\partial_3^3\omega_1 dx-\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\omega_1\partial_3^3\omega_1 dxd\tau\\
&\qquad+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\partial_1 u_3\partial_3^3\omega_1 dx+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\partial_2 u_3\partial_3^3\omega_1 dxd\tau.
\end{aligned}\tag{3.7}
$$

The first two terms on the right-hand side of (3.7) can be handled by Proposition 1.1, while the remaining two terms can be handled by Sobolev's inequality:

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\omega_2\partial_3^3\omega_1 dx-\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\omega_1\partial_3^3\omega_1 dxd\tau\lesssim\mathscr{E}^{\frac{3}{2}}(0)+\mathscr{E}^{\frac{3}{2}}(t)+\mathscr{E}^2(t),\\
&\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\partial_1 u_3\partial_3^3\omega_1 dx+\int_0^t\int_{\mathbb{R}^3}\partial_2 u_1\partial_3^3\partial_2 u_3\partial_3^3\omega_1 dxd\tau\lesssim\mathscr{E}^{\frac{3}{2}}(t).
\end{aligned}
$$

Taking all the inequalities above into consideration, we then obtain

$$
\int_0^t|I_2(\tau)|d\tau\lesssim\mathscr{E}^{\frac{3}{2}}(0)+\mathscr{E}^{\frac{3}{2}}(t)+\mathscr{E}^2(t).\tag{3.8}
$$

We shall combine the estimates of I_3 and I_7 to take advantage of cancellations. Naturally, $I_3=\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i^3(b\cdot\nabla H)\cdot\partial_i^3\omega dx$ is divided into the following three terms:

$$
\int_{\mathbb{R}^3}\partial_1^3(b\cdot\nabla H)\cdot\partial_1^3\omega dx,\quad\int_{\mathbb{R}^3}\partial_2^3(b\cdot\nabla H)\cdot\partial_2^3\omega dx,\quad\text{and}\int_{\mathbb{R}^3}\partial_3^3(b\cdot\nabla H)\cdot\partial_3^3\omega dx.
$$

Each term above can be split into two parts as follows:

$$
I_{31}+\tilde{I}_{31}=\sum_{k=1}^3\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_1^k b\cdot\nabla\partial_1^{3-k}H\cdot\partial_1^3\omega dx+\int_{\mathbb{R}^3}b\cdot\nabla\partial_1^3 H\cdot\partial_1^3\omega dx,
$$

$$
I_{32}+\tilde{I}_{32}=\sum_{k=1}^3\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_2^k b\cdot\nabla\partial_2^{3-k}H\cdot\partial_2^3\omega dx+\int_{\mathbb{R}^3}b\cdot\nabla\partial_2^3 H\cdot\partial_2^3\omega dx,
$$

$$I_{33} + \tilde{I}_{33} = \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k b \cdot \nabla \partial_3^{3-k} H \cdot \partial_3^3 \omega dx + \int_{\mathbb{R}^3} b \cdot \nabla \partial_3^3 H \cdot \partial_3^3 \omega dx.$$

I_{31} and I_{32} have the good derivatives ∇_h and can be bounded directly:

$$I_{31} + I_{32} \lesssim \|\nabla_h b\|_{H^2} \|b\|_{H^4} \|\nabla_h \omega\|_{H^2}.$$

To deal with I_{33}, we also use the divergence-free property $\nabla \cdot b = 0$ to write

$$I_{33} = \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k b_h \cdot \nabla_h \partial_3^{3-k} H \cdot \partial_3^3 \omega dx - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^{k-1} \nabla_h \cdot b_h \partial_3^{4-k} H \cdot \partial_3^3 \omega dx.$$

We have converted some of ∂_3 into the good derivatives ∇_h. By Lemma 2.1,

$$\begin{aligned} I_{33} \lesssim \Big(& \|b_h\|_{H^3}^{\frac{1}{2}} \|\partial_{\mathbf{2}} b_h\|_{H^3}^{\frac{1}{2}} \|\nabla_h H\|_{H^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla_h H\|_{H^2}^{\frac{1}{2}} + \|\nabla_h b\|_{H^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla_h b\|_{H^2}^{\frac{1}{2}} \|H\|_{H^3}^{\frac{1}{2}} \|\partial_{\mathbf{2}} H\|_{H^3}^{\frac{1}{2}} \Big) \\ & \cdot \|\partial_3^3 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{1}} \partial_3^3 \omega\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

The remaining terms $\tilde{I}_{31}$, $\tilde{I}_{32}$, and $\tilde{I}_{33}$ cannot be controlled directly but will be canceled by the corresponding terms in I_7. Now let's focus on $I_7 = \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 [\nabla \times (b \cdot \nabla u)] \cdot \partial_i^3 H dx$. We notice that

$$\nabla \times (b \cdot \nabla u) = b \cdot \nabla \omega + \mathcal{R},$$

where $\mathcal{R}$ stands for the vector with its ith component given by

$$\mathcal{R}_i = \sigma_{ijk} \partial_j b \cdot \nabla u_k.$$

Here σ_{ijk} is the Levi–Cevita symbol:

$$\sigma_{ijk} = \begin{cases} 1, & ijk = 123, 231, 312, \\ -1, & ijk = 321, 213, 132, \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$

Following the process for I_3, we can split I_7 into the following nine parts:

$$\begin{aligned} & I_{71} + \tilde{I}_{71} + \tilde{\tilde{I}}_{71} \\ & \quad = \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_1^k b \cdot \nabla \partial_1^{3-k} \omega \cdot \partial_1^3 H dx + \int_{\mathbb{R}^3} b \cdot \nabla \partial_1^3 \omega \cdot \partial_1^3 H dx + \int_{\mathbb{R}^3} \partial_1^3 (\mathcal{R}) \cdot \partial_1^3 H dx, \end{aligned}$$

$$\begin{aligned} & I_{72} + \tilde{I}_{72} + \tilde{\tilde{I}}_{72} \\ & \quad = \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b \cdot \nabla \partial_2^{3-k} \omega \cdot \partial_2^3 H dx + \int_{\mathbb{R}^3} b \cdot \nabla \partial_2^3 \omega \cdot \partial_2^3 H dx + \int_{\mathbb{R}^3} \partial_2^3 (\mathcal{R}) \cdot \partial_2^3 H dx, \end{aligned}$$

$$\begin{aligned} & I_{73} + \tilde{I}_{73} + \tilde{\tilde{I}}_{73} \\ & \quad = \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k b \cdot \nabla \partial_3^{3-k} \omega \cdot \partial_3^3 H dx + \int_{\mathbb{R}^3} b \cdot \nabla \partial_3^3 \omega \cdot \partial_3^3 H dx + \int_{\mathbb{R}^3} \partial_3^3 (\mathcal{R}) \cdot \partial_3^3 H dx. \end{aligned}$$

By integration by parts and $\nabla\cdot b=0$,

$$\tilde{I}_{31}+\tilde{I}_{71}=0,\quad \tilde{I}_{32}+\tilde{I}_{72}=0,\quad \text{and}\quad \tilde{I}_{33}+\tilde{I}_{73}=0.$$

I_{71} and I_{72} are easy to control, while I_{73} can be written as

$$I_{73}=\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k b_h\cdot\nabla_h\partial_3^{3-k}\omega\cdot\partial_3^3H\,dx-\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^{k-1}\nabla_h\cdot b_h\partial_3^{4-k}\omega\cdot\partial_3^3H\,dx.$$

Now I_{71},I_{72}, and I_{73} all contain good derivatives ∇_h and can be handled exactly like I_{31},I_{32}, and I_{33}. We omit the repeated details. Let's focus on the remaining terms containing $\mathcal{R}$, namely, $\tilde{\tilde{I}}_{71}$, $\tilde{\tilde{I}}_{72}$, and $\tilde{\tilde{I}}_{73}$. By Hölder's inequality and the Sobolev embedding theorem,

$$\tilde{\tilde{I}}_{71}+\tilde{\tilde{I}}_{72}\lesssim\big(\|\nabla_h b\|_{H^3}\|u\|_{H^4}+\|\nabla_h u\|_{H^3}\|b\|_{H^4}\big)\|\nabla_h H\|_{H^3}.$$

$\tilde{\tilde{I}}_{73}$ is more complex and is further split into two parts:

$$\begin{aligned}\tilde{\tilde{I}}_{73}=&\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_3^3(\sigma_{ijk}\partial_j b_h\cdot\nabla_h u_k)\partial_3^3H_i\,dx+\int_{\mathbb{R}^3}\partial_3^3(\sigma_{3jk}\partial_j b_3\partial_3u_k)\partial_3^3H_3\,dx\\&+\sum_{i\neq3}\int_{\mathbb{R}^3}\partial_3^3(\sigma_{ijk}\partial_j b_3\partial_3u_k)\partial_3^3H_i\,dx=\tilde{\tilde{I}}_{731}+\tilde{\tilde{I}}_{732}.\end{aligned}$$

Noticing $H_3=\partial_1b_2-\partial_2b_1$ and applying Lemma 2.1, we can bound $\tilde{\tilde{I}}_{731}$ by

$$\begin{aligned}\tilde{\tilde{I}}_{731}\lesssim&\|\nabla_hu\|_{H^3}\|b_h\|_{H^3}^{\frac14}\|\partial_{\mathbf{2}}b_h\|_{H^3}^{\frac14}\|\partial_{\mathbf{3}}b_h\|_{H^3}^{\frac14}\|\partial_{\mathbf{2}}\partial_{\mathbf{3}}b_h\|_{H^3}^{\frac14}\|\partial_3^3H\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3H\|_{L^2}^{\frac12}\\&+\|\nabla_hu\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\nabla_hu\|_{L^2}^{\frac12}\|b_h\|_{H^4}^{\frac12}\|\partial_{\mathbf{2}}b_h\|_{H^4}^{\frac12}\|\partial_3^3H\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3H\|_{L^2}^{\frac12}\\&+\|b_3\|_{H^4}^{\frac12}\|\partial_{\mathbf{2}}b_3\|_{H^4}^{\frac12}\|u\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}}u\|_{H^4}^{\frac12}\|\nabla_hb\|_{H^3}^{\frac12}\|\partial_{\mathbf{3}}\nabla_hb\|_{H^3}^{\frac12}.\end{aligned}$$

By the definition of σ_{ijk}, if $i\neq3$, we will have $j=3$ or $k=3$. Indeed,

$$\begin{aligned}\tilde{\tilde{I}}_{732}\lesssim&\|\partial_3b_3\|_{H^3}^{\frac12}\|\partial_{\mathbf{3}}\partial_3b_3\|_{H^3}^{\frac12}\|u\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}}u\|_{H^4}^{\frac12}\|\partial_3^3H\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^3H\|_{L^2}^{\frac12}\\&+\|\partial_3u_3\|_{H^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_3u_3\|_{H^2}^{\frac12}\|b_3\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}}b_3\|_{H^4}^{\frac12}\|\partial_3^3H\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^3H\|_{L^2}^{\frac12}\\&+\|\partial_3u_3\|_{H^3}\|b_3\|_{H^1}^{\frac14}\|\partial_{\mathbf{1}}b_3\|_{H^1}^{\frac14}\|\partial_{\mathbf{3}}b_3\|_{H^1}^{\frac14}\|\partial_{\mathbf{1}}\partial_{\mathbf{3}}b_3\|_{H^1}^{\frac14}\|\partial_3^3H\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^3H\|_{L^2}^{\frac12}.\end{aligned}$$

Collecting the upper bounds for I_3 and I_7 above, we find

$$\int_0^t|I_3(\tau)+I_7(\tau)|d\tau\lesssim\mathscr{E}^{\frac32}(t).\tag{3.10}$$

Next we deal with I_4, which is naturally divided into the following three terms:

$$-\int_{\mathbb{R}^3}\partial_1^3(H\cdot\nabla b)\cdot\partial_1^3\omega\,dx,\quad-\int_{\mathbb{R}^3}\partial_2^3(H\cdot\nabla b)\cdot\partial_2^3\omega\,dx,\quad-\int_{\mathbb{R}^3}\partial_3^3(H\cdot\nabla b)\cdot\partial_3^3\omega\,dx.$$

The first two terms already contain the good derivatives ∇_h and can be bounded directly. The third term is further decomposed into three terms:

$$\begin{aligned}-\int_{\mathbb{R}^3}\partial_3^3(H\cdot\nabla b)\cdot\partial_3^3\omega\,dx=&-\int_{\mathbb{R}^3}\partial_3^3(H_1\partial_1b)\cdot\partial_3^3\omega\,dx-\int_{\mathbb{R}^3}\partial_3^3(H_2\partial_2b)\cdot\partial_3^3\omega\,dx\\&-\int_{\mathbb{R}^3}\partial_3^3(H_3\partial_3b)\cdot\partial_3^3\omega\,dx.\end{aligned}$$

The first two terms above have either $\partial_1 b$ or $\partial_2 b$ and can thus be bounded by applying Lemma 2.1. The third term involves $H_3 = \partial_1 b_2 - \partial_2 b_1$ and thus also contains the good horizontal derivatives on b. Therefore, all of them admit suitable upper bounds and

$$\int_0^t |I_4(\tau)| d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(t). \tag{3.11}$$

I_5 and I_8 cancel each other by integration by parts:

$$I_5 + I_8 = \sum_{i=1}^{3} \left[\int_{\mathbb{R}^3} \partial_i^3 \partial_2 H \cdot \partial_i^3 \omega dx + \int_{\mathbb{R}^3} \partial_i^3 \partial_2 \omega \cdot \partial_i^3 H dx \right] = 0. \tag{3.12}$$

It remains to deal with I_6. As in I_7, we can write $\nabla \times (u \cdot \nabla b) = u \cdot \nabla H + \tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}}_i = \sigma_{ijk} \partial_j u \cdot \nabla b_k$ and σ_{ijk} is defined in (3.9). Thus,

$$\begin{aligned}
I_6 = & - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_1^k u \cdot \nabla \partial_1^{3-k} H \cdot \partial_1^3 H dx - \int_{\mathbb{R}^3} \partial_1^3 (\tilde{\mathcal{R}}) \cdot \partial_1^3 H dx \\
& - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u \cdot \nabla \partial_2^{3-k} H \cdot \partial_2^3 H dx - \int_{\mathbb{R}^3} \partial_2^3 (\tilde{\mathcal{R}}) \cdot \partial_2^3 H dx \\
& - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k u_h \cdot \nabla_h \partial_3^{3-k} H \cdot \partial_3^3 H dx - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k u_3 \partial_3^{4-k} H \cdot \partial_3^3 H dx \\
& - \int_{\mathbb{R}^3} \partial_3^3 (\tilde{\mathcal{R}}) \cdot \partial_3^3 H dx \\
= & \, I_{61} + I_{62} + I_{63} + I_{64}.
\end{aligned}$$

By Hölder's inequality and the Sobolev embedding theorem,

$$\begin{aligned}
I_{61} + I_{62} \lesssim & \|\nabla_h u\|_{H^3} \|H\|_{H^3} \|\nabla_h H\|_{H^3} \\
& + \big(\|\nabla_h u\|_{H^3} \|b\|_{H^4} + \|u\|_{H^2} \|\nabla_h b\|_{H^3} \big) \|\nabla_h H\|_{H^3}.
\end{aligned}$$

The estimate for I_{63} is similar to that for I_{73}:

$$\begin{aligned}
I_{63} \lesssim & \|u_h\|_{H^3}^{\frac{1}{2}} \|\partial_{\mathbf{1}} u_h\|_{H^3}^{\frac{1}{2}} \|\nabla_h H\|_{H^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla_h H\|_{H^2}^{\frac{1}{2}} \|\partial_3^3 H\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_3^3 H\|_{L^2}^{\frac{1}{2}} \\
& + \|\nabla_h \cdot u_h\|_{H^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla_h \cdot u_h\|_{H^2}^{\frac{1}{2}} \|H\|_{H^3} \|\nabla_{\mathbf{h}} H\|_{H^3}.
\end{aligned}$$

I_{64} is bounded similarly as $\tilde{\tilde{I}}_{73}$:

$$\begin{aligned}
I_{64} \lesssim & \|u_h\|_{H^4}^{\frac{1}{2}} \|\partial_{\mathbf{1}} u_h\|_{H^4}^{\frac{1}{2}} \|\nabla_h b\|_{H^3}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla_h b\|_{H^3}^{\frac{1}{2}} \|\partial_3^3 H\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_3^3 H\|_{L^2}^{\frac{1}{2}} \\
& + \|u_3\|_{H^4}^{\frac{1}{2}} \|\partial_{\mathbf{1}} u_3\|_{H^4}^{\frac{1}{2}} \|b\|_{H^4}^{\frac{1}{2}} \|\partial_{\mathbf{2}} b\|_{H^4}^{\frac{1}{2}} \|\nabla_h b\|_{H^3}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \nabla_h b\|_{H^3}^{\frac{1}{2}} \\
& + \|\partial_3 b_3\|_{H^3}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \partial_3 b_3\|_{H^3}^{\frac{1}{2}} \|u\|_{H^4}^{\frac{1}{2}} \|\partial_{\mathbf{1}} u\|_{H^4}^{\frac{1}{2}} \|\partial_3^3 H\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_3^3 H\|_{L^2}^{\frac{1}{2}} \\
& + \|\partial_3 u_3\|_{H^2}^{\frac{1}{2}} \|\partial_{\mathbf{3}} \partial_3 u_3\|_{H^2}^{\frac{1}{2}} \|b\|_{H^4}^{\frac{1}{2}} \|\partial_{\mathbf{1}} b\|_{H^4}^{\frac{1}{2}} \|\partial_3^3 H\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_3^3 H\|_{L^2}^{\frac{1}{2}} \\
& + \|\partial_3 u_3\|_{H^3} \|b\|_{H^1}^{\frac{1}{4}} \|\partial_{\mathbf{1}} b\|_{H^1}^{\frac{1}{4}} \|\partial_{\mathbf{3}} b\|_{H^1}^{\frac{1}{4}} \|\partial_{\mathbf{1}} \partial_{\mathbf{3}} b\|_{H^1}^{\frac{1}{4}} \|\partial_3^3 H\|_{L^2}^{\frac{1}{2}} \|\partial_{\mathbf{2}} \partial_3^3 H\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\int_0^t |I_6(\tau)| d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(t). \tag{3.13}$$

Integrating (3.3) in time on the interval $[0,t]$ and invoking the upper bounds in (3.4), (3.8), (3.10), (3.11), (3.12), and (3.13), we obtain

$$\sup_{0\le\tau\le t}\left(\|u\|_{\dot H^4}^2+\|b\|_{\dot H^4}^2\right)+\int_0^t\left(\|\partial_1 u\|_{\dot H^4}^2+\|\nabla_h b\|_{\dot H^4}^2\right)d\tau\lesssim \mathscr{E}(0)+\mathscr{E}^{\frac32}(0)+\mathscr{E}^{\frac32}(t)+\mathscr{E}^2(t). \tag{3.14}$$

Adding (3.1) and (3.14) yields the desired inequality in (1.11), namely,

$$\mathscr{E}_1(t)\lesssim \mathscr{E}(0)+\mathscr{E}^{\frac32}(0)+\mathscr{E}^{\frac32}(t)+\mathscr{E}^2(t).$$

This finishes the proof. ◻

4. Proof of Proposition 1.1.

This section is devoted to the proof of Proposition 1.1, which provides suitable upper bounds for the time integral of the interaction terms. We have used Proposition 1.1 extensively in the crucial energy estimates in the previous sections.

The proof of Proposition 1.1 deals with some of the most difficult terms emanating from the velocity nonlinearity. We take out two of the wildest terms and deal with them in the following lemma. We will state and prove this lemma and then prove Proposition 1.1.

LEMMA 4.1. *Let $(u,b)\in H^4$ be a solution of* (1.2). *Let $\omega=\nabla\times u$ and $H=\nabla\times b$ be the corresponding vorticity and current density, respectively. Let $\mathscr{E}(t)$ be defined as in* (1.8). *Then, for all $i,j,k\in\{1,2,3\}$,*

$$\left|\int_0^t\int_{\mathbb{R}^3} b_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)\,dxd\tau\right|,\ \left|\int_0^t\int_{\mathbb{R}^3}\partial_2 u_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)\,dxd\tau\right|\lesssim \mathscr{E}^{\frac32}(0)+\mathscr{E}^{\frac32}(t)+\mathscr{E}^2(t).$$

Proof of Lemma 4.1. We will deal with the two terms above simultaneously. By the equation of ω in (3.2),

$$\begin{aligned}
&-\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)\\
&=\partial_3^3\omega_i\partial_3^3\big(u\cdot\nabla\omega_k-\omega\cdot\nabla u_k-\partial_1^2\omega_k-b\cdot\nabla H_k+H\cdot\nabla b_k-\partial_2 H_k\big)\\
&\quad+\partial_3^3\omega_k\partial_3^3\big(u\cdot\nabla\omega_i-\omega\cdot\nabla u_i-\partial_1^2\omega_i-b\cdot\nabla H_i+H\cdot\nabla b_i-\partial_2 H_i\big).
\end{aligned}\tag{4.1}$$

Multiplying (4.1) by b_j or $\partial_2 u_j$ and integrating in space, we can write

$$-\int_{\mathbb{R}^3}\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)(b_j|\partial_2 u_j)dx=J_1+J_2+J_3+J_4+J_5,\tag{4.2}$$

where the notation $(b_j|\partial_2 u_j)$ stands for either b_j or $\partial_2 u_j$, and we will use it throughout the rest of the proof. The explicit expression for $J_1\sim J_5$ is shown below. We first deal with J_1 given by

$$J_1=\int_{\mathbb{R}^3}\left[\partial_3^3\omega_i\partial_3^3(u\cdot\nabla\omega_k)+\partial_3^3\omega_k\partial_3^3(u\cdot\nabla\omega_i)\right](b_j|\partial_2 u_j)dx.$$

The highest-order norms of ω in J_1, labeled as J_{11}, can be dealt with using Lemma 2.1:

$$\begin{aligned}
J_{11}&=\int_{\mathbb{R}^3}\Big[\partial_3^3\omega_i u\cdot\nabla\partial_3^3\omega_k+\partial_3^3\omega_k u\cdot\nabla\partial_3^3\omega_i\Big](b_j|\partial_2 u_j)dx\\
&=\int_{\mathbb{R}^3}u\cdot\nabla\big(\partial_3^3\omega_i\partial_3^3\omega_k\big)(b_j|\partial_2 u_j)dx\\
&=-\int_{\mathbb{R}^3}\partial_3^3\omega_i\partial_3^3\omega_k u\cdot\nabla(b_j|\partial_2 u_j)dx\\
&\lesssim\|\boldsymbol{\partial_1}\omega\|_{H^3}\|\omega\|_{H^3}\|\boldsymbol{\partial_2} u\|_{H^1}^{\frac12}\|u\|_{H^1}^{\frac12}\|\boldsymbol{\partial_2}(b|\partial_2 u)\|_{H^2}^{\frac12}\|(b|\partial_2 u)\|_{H^2}^{\frac12}.
\end{aligned}$$

The remaining parts in J_1 can be written as

$$\begin{aligned}
J_{12} =& \sum_{l=1}^{3} \mathcal{C}_3^l \int_{\mathbb{R}^3} \left[\partial_3^3\omega_i \partial_3^l u \cdot \nabla \partial_3^{3-l}\omega_k + \partial_3^3\omega_k \partial_3^l u \cdot \nabla \partial_3^{3-l}\omega_i\right] (b_j|\partial_2 u_j) dx \\
=& \sum_{l=1}^{2} \mathcal{C}_3^l \int_{\mathbb{R}^3} \left[\partial_3^3\omega_i \partial_3^l u \cdot \nabla \partial_3^{3-l}\omega_k + \partial_3^3\omega_k \partial_3^l u \cdot \nabla \partial_3^{3-l}\omega_i\right] (b_j|\partial_2 u_j) dx \\
&+ \int_{\mathbb{R}^3} \left[\partial_3^3\omega_i \partial_3^3 u \cdot \nabla \omega_k + \partial_3^3\omega_k \partial_3^3 u \cdot \nabla \omega_i\right] (b_j|\partial_2 u_j) dx,
\end{aligned}$$

which is easily controlled by

$$\|\partial_3^3\omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12} \|u\|_{H^4}^{\frac12} \|\partial_{\mathbf{1}} u\|_{H^4}^{\frac12} \|u\|_{H^3}^{\frac12} \|\partial_{\mathbf{2}} u\|_{H^3}^{\frac12} \|(b|\partial_2 u)\|_{H^1}^{\frac12} \|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{H^1}^{\frac12}.$$

J_2 is defined as and bounded by

$$\begin{aligned}
J_2 =& -\int_{\mathbb{R}^3} \partial_3^3\omega_i \partial_3^3(\omega\cdot\nabla u_k)(b_j|\partial_2 u_j) dx - \int_{\mathbb{R}^3} \partial_3^3\omega_k \partial_3^3(\omega\cdot\nabla u_i)(b_j|\partial_2 u_j) dx \\
\lesssim& \|\partial_3^3\omega\|_{L^2}^{\frac12} \|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12} \|u\|_{H^4}^{\frac12} \|\partial_{\mathbf{1}} u\|_{H^4}^{\frac12} \|u\|_{H^2}^{\frac14} \|\partial_{\mathbf{2}} u\|_{H^2}^{\frac14} \|\partial_{\mathbf{3}} u\|_{H^2}^{\frac14} \|\partial_{\mathbf{2}}\partial_{\mathbf{3}} u\|_{H^2}^{\frac14} \\
&\cdot \|(b|\partial_2 u)\|_{L^2}^{\frac14} \|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{L^2}^{\frac14} \|\partial_{\mathbf{3}}(b|\partial_2 u)\|_{L^2}^{\frac14} \|\partial_{\mathbf{2}}\partial_{\mathbf{3}}(b|\partial_2 u)\|_{L^2}^{\frac14}.
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
J_3 =& \int_{\mathbb{R}^3} \left[-\partial_3^3\omega_i \partial_3^3\partial_1^2\omega_k - \partial_3^3\omega_k \partial_3^3\partial_1^2\omega_i\right] (b_j|\partial_2 u_j)\, dx \\
=& 2\int_{\mathbb{R}^3} \partial_1\partial_3^3\omega_i \partial_1\partial_3^3\omega_k (b_j|\partial_2 u_j) dx \\
&+ \int_{\mathbb{R}^3} \left[\partial_3^3\omega_i \partial_1\partial_3^3\omega_k + \partial_3^3\omega_k \partial_1\partial_3^3\omega_i\right] \partial_1(b_j|\partial_2 u_j) dx \\
\lesssim& \|\partial_1\omega\|_{H^3}^2 \|(b|\partial_2 u)\|_{H^2} + \|\omega\|_{H^3} \|\partial_1\omega\|_{H^3} \|\partial_1(b|\partial_2 u)\|_{H^2}.
\end{aligned}$$

J_4 is given by

$$J_4 = \int_{\mathbb{R}^3} \left[-\partial_3^3\omega_i \partial_3^3(b\cdot\nabla H_k) - \partial_3^3\omega_k \partial_3^3(b\cdot\nabla H_i)\right] (b_j|\partial_2 u_j)\, dx.$$

This is a difficult term. First, we separate J_4 into two parts:

$$\begin{aligned}
J_4 =& \int_{\mathbb{R}^3} \left[-\partial_3^3\omega_i b\cdot\nabla\partial_3^3 H_k - \partial_3^3\omega_k b\cdot\nabla\partial_3^3 H_i\right] (b_j|\partial_2 u_j)\, dx \\
&+ \sum_{l=1}^{3} \mathcal{C}_3^l \int_{\mathbb{R}^3} \left[-\partial_3^3\omega_i \partial_3^l b\cdot\nabla\partial_3^{3-l} H_k - \partial_3^3\omega_k \partial_3^l b\cdot\nabla\partial_3^{3-l} H_i\right] (b_j|\partial_2 u_j) dx \\
=& J_{41} + J_{42}.
\end{aligned}$$

J_{42} can be bounded directly. By Lemma 2.1,

$$\begin{aligned}
|J_{42}| \lesssim& \|\omega\|_{H^3}^{\frac12} \|\partial_{\mathbf{1}}\omega\|_{H^3}^{\frac12} \|H\|_{H^3}^{\frac12} \|\partial_{\mathbf{1}} H\|_{H^3}^{\frac12} \|b\|_{H^4}^{\frac12} \|\partial_{\mathbf{2}} b\|_{H^4}^{\frac12} \\
&\cdot \|(b|\partial_2 u)\|_{H^1}^{\frac12} \|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{H^1}^{\frac12}.
\end{aligned} \tag{4.3}$$

The estimate for J_{41} is at the core of this section. J_1, J_2, and J_3 are symmetric in the sense that, when we switch i and k in any one of these terms, they remain the same. As we have seen in the estimates above, terms with symmetric structure are relatively easy to deal with. However, J_{41} is not symmetric, and we can no longer make easy cancellations. To overcome this essential difficulty, we construct some artificial symmetry to take full advantage of the cancellations:

$$\begin{aligned}
J_{41} &= \int_{\mathbb{R}^3} \big[-\partial_3^3\omega_i b\cdot\nabla\partial_3^3 H_k - \partial_3^3\omega_k b\cdot\nabla\partial_3^3 H_i\big](b_j|\partial_2 u_j)\,dx \\
&= \int_{\mathbb{R}^3} \big[\partial_3^3 H_i b\cdot\nabla\partial_3^3\omega_k + \partial_3^3 H_k b\cdot\nabla\partial_3^3\omega_i\big](b_j|\partial_2 u_j)\,dx \\
&\quad - \int_{\mathbb{R}^3} \big[b\cdot\nabla(\partial_3^3\omega_i\partial_3^3 H_k) + b\cdot\nabla(\partial_3^3\omega_k\partial_3^3 H_i)\big](b_j|\partial_2 u_j)\,dx \\
&= J_{411} + J_{412}.
\end{aligned}\tag{4.4}$$

J_{412} can be handled through integration by parts and Lemma 2.1:

$$\begin{aligned}
J_{412} &= \int_{\mathbb{R}^3} \big[\partial_3^3\omega_i\partial_3^3 H_k + \partial_3^3\omega_k\partial_3^3 H_i\big]\, b\cdot\nabla(b_j|\partial_2 u_j)dx \\
&\lesssim \|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_3^3 H\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3 H\|_{L^2}^{\frac12}\|b\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}} b\|_{H^1}^{\frac12} \\
&\qquad \cdot\|(b|\partial_2 u)\|_{H^2}^{\frac12}\|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{H^2}^{\frac12}.
\end{aligned}\tag{4.5}$$

J_{411} is extremely difficult. As our first step, we invoke the equation of H in (3.2),

$$\begin{aligned}
&\partial_t H_k + u\cdot\nabla H_k + \sum_{p=1}^{3}\nabla u_p\times\partial_p b_k - \Delta_h H_k \\
&= b\cdot\nabla\omega_k + \sum_{p=1}^{3}\nabla b_p\times\partial_p u_k + \partial_2\omega_k,
\end{aligned}\tag{4.6}$$

where we have used the simple identities

$$\begin{aligned}
\nabla\times(b\cdot\nabla u) &= b\cdot\nabla\omega + \sum_{p=1}^{3}\nabla b_p\times\partial_p u, \\
\nabla\times(u\cdot\nabla b) &= u\cdot\nabla H + \sum_{p=1}^{3}\nabla u_p\times\partial_p b.
\end{aligned}$$

Applying ∂_3^3 to (4.6) then yields

$$\begin{aligned}
&(\partial_3^3 H_k)_t + \partial_3^3(u\cdot\nabla H_k) + \partial_3^3\left(\sum_{p=1}^{3}\nabla u_p\times\partial_p b_k\right) - \partial_3^3\Delta_h H_k \\
&= b\cdot\nabla\partial_3^3\omega_k + \sum_{l=1}^{3}\mathcal{C}_3^l\partial_3^l b\cdot\nabla\partial_3^{3-l}\omega_k + \partial_3^3\left(\sum_{p=1}^{3}\nabla b_p\times\partial_p u_k\right) + \partial_3^3\partial_2\omega_k.
\end{aligned}\tag{4.7}$$

Similarly,

$$\begin{aligned}
&(\partial_3^3 H_i)_t + \partial_3^3(u\cdot\nabla H_i) + \partial_3^3\left(\sum_{p=1}^{3}\nabla u_p\times\partial_p b_i\right) - \partial_3^3\Delta_h H_i \\
&= b\cdot\nabla\partial_3^3\omega_i + \sum_{l=1}^{3}\mathcal{C}_3^l\partial_3^l b\cdot\nabla\partial_3^{3-l}\omega_i + \partial_3^3\left(\sum_{p=1}^{3}\nabla b_p\times\partial_p u_i\right) + \partial_3^3\partial_2\omega_i.
\end{aligned}\tag{4.8}$$

Multiplying (4.7) by $\partial_3^3 H_i$ and (4.8) by $\partial_3^3 H_k$ and summing them up, there holds

$$\begin{aligned}
&\partial_3^3 H_i b\cdot\nabla\partial_3^3\omega_k + \partial_3^3 H_k b\cdot\nabla\partial_3^3\omega_i\\
&\quad= \big(\partial_3^3 H_i\partial_3^3 H_k\big)_t + \partial_3^3 H_i\partial_3^3(u\cdot\nabla H_k) + \partial_3^3 H_k\partial_3^3(u\cdot\nabla H_i)\\
&\qquad+ \partial_3^3 H_i\partial_3^3\left(\sum_{p=1}^{3}\nabla u_p\times\partial_p b_k\right) + \partial_3^3 H_k\partial_3^3\left(\sum_{p=1}^{3}\nabla u_p\times\partial_p b_i\right)\\
&\qquad- \partial_3^3 H_i\partial_3^3\left(\sum_{p=1}^{3}\nabla b_p\times\partial_p u_k\right) - \partial_3^3 H_k\partial_3^3\left(\sum_{p=1}^{3}\nabla b_p\times\partial_p u_i\right)\\
&\qquad- \partial_3^3 H_i\sum_{l=1}^{3}\mathcal{C}_3^l\partial_3^l b\cdot\nabla\partial_3^{3-l}\omega_k - \partial_3^3 H_k\sum_{l=1}^{3}\mathcal{C}_3^l\partial_3^l b\cdot\nabla\partial_3^{3-l}\omega_i\\
&\qquad- \partial_3^3 H_i\partial_3^3\Delta_h H_k - \partial_3^3 H_k\partial_3^3\Delta_h H_i - \partial_3^3 H_i\partial_3^3\partial_2\omega_k - \partial_3^3 H_k\partial_3^3\partial_2\omega_i.
\end{aligned}\tag{4.9}$$

We can then replace the nonlinear terms in J_{411} by the right-hand side of (4.9). This complicated substitution generates many more terms, but this important process converts some of seemingly impossible terms into other terms that can be bounded suitably. This is what we would call artificial cancellation through substitutions. Next we give the estimate for the terms corresponding to the right-hand side of (4.9). The first term we come across is

$$K_1 = \int_{\mathbb{R}^3}\big(\partial_3^3 H_i\partial_3^3 H_k\big)_t(b_j|\partial_2 u_j)\,dx.$$

Using integration by parts and invoking the equation of b in (1.2), we can rewrite the term containing b_j as follows:

$$\begin{aligned}
K_1 &= \frac{d}{dt}\int_{\mathbb{R}^3}\big(\partial_3^3 H_i\partial_3^3 H_k\big)b_j\,dx - \int_{\mathbb{R}^3}\big(\partial_3^3 H_i\partial_3^3 H_k\big)\partial_t b_j\,dx\\
&= \frac{d}{dt}\int_{\mathbb{R}^3}\big(\partial_3^3 H_i\partial_3^3 H_k\big)b_j\,dx\\
&\quad- \int_{\mathbb{R}^3}\big(\partial_3^3 H_i\partial_3^3 H_k\big)(-u\cdot\nabla b_j + \Delta_h b_j + b\cdot\nabla u_j + \partial_2 u_j)\,dx.
\end{aligned}$$

By Lemma 2.1, the last line in the equality above can be bounded by

$$\begin{aligned}
&\|\partial_3^3 H\|_{L^2}\|\boldsymbol{\partial_1}\partial_3^3 H\|_{L^2}\|u\|_{H^2}^{\frac12}\|\boldsymbol{\partial_2}u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12}\|\boldsymbol{\partial_2}b\|_{H^2}^{\frac12}\\
&\quad+ \|\partial_3^3 H\|_{L^2}^{\frac12}\|\boldsymbol{\partial_1}\partial_3^3 H\|_{L^2}^{\frac12}\|\partial_3^3 H\|_{L^2}^{\frac12}\|\boldsymbol{\partial_2}\partial_3^3 H\|_{L^2}^{\frac12}\\
&\qquad\times\left(\|\Delta_h b\|_{L^2}^{\frac12}\|\boldsymbol{\partial_3}\Delta_h b\|_{L^2}^{\frac12} + \|\partial_2 u\|_{L^2}^{\frac12}\|\boldsymbol{\partial_3}\partial_2 u\|_{L^2}^{\frac12}\right).
\end{aligned}$$

The estimate for the term containing $\partial_2 u_j$ is similar. We move on to the second part:

$$K_2 = \int_{\mathbb{R}^3}\big[\partial_3^3 H_i\partial_3^3(u\cdot\nabla H_k) + \partial_3^3 H_k\partial_3^3(u\cdot\nabla H_i)\big](b_j|\partial_2 u_j)\,dx.$$

It can be handled similarly as J_{11}. Due to its symmetric property,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Big[\partial_3^3 H_i u\cdot\nabla\partial_3^3 H_k + \partial_3^3 H_k u\cdot\nabla\partial_3^3 H_i\Big](b_j|\partial_2 u_j)dx\\
&\quad= \int_{\mathbb{R}^3}u\cdot\nabla\big(\partial_3^3 H_i\partial_3^3 H_k\big)(b_j|\partial_2 u_j)dx
\end{aligned}$$

$$\begin{aligned}
&= -\int_{\mathbb{R}^3} \partial_3^3 H_i \partial_3^3 H_k u \cdot \nabla(b_j|\partial_2 u_j)dx \\
&\lesssim \|\partial_{\mathbf{1}} H\|_{H^3}\|H\|_{H^3}\|\partial_{\mathbf{2}} u\|_{H^1}^{\frac12}\|u\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{H^2}^{\frac12}\|(b|\partial_2 u)\|_{H^2}^{\frac12}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{l=1}^{3}\mathcal{C}_3^l\int_{\mathbb{R}^3}\Big[\partial_3^3 H_i \partial_3^l u\cdot\nabla\partial_3^{3-l}H_k + \partial_3^3 H_k\partial_3^l u\cdot\nabla\partial_3^{3-l}H_i\Big](b_j|\partial_2 u_j)dx \\
&\lesssim \|\partial_3^3 H\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^3 H\|_{L^2}^{\frac12}\|H\|_{H^3}^{\frac12}\|\partial_{\mathbf{2}} H\|_{H^3}^{\frac12}\|u\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}} u\|_{H^4}^{\frac12}\|(b|\partial_2 u)\|_{H^1}^{\frac12}\|\partial_{\mathbf{1}}(b|\partial_2 u)\|_{H^1}^{\frac12}.
\end{aligned}$$

The next term K_3 contains six parts of (4.9) and is defined as follows:

$$\begin{aligned}
K_3 = \int_{\mathbb{R}^3}\Bigg(&\partial_3^3 H_i\partial_3^3\left(\sum_{p=1}^{3}\nabla u_p\times\partial_p b_k\right) + \partial_3^3 H_k\partial_3^3\left(\sum_{p=1}^{3}\nabla u_p\times\partial_p b_i\right) \\
&-\partial_3^3 H_i\partial_3^3\left(\sum_{p=1}^{3}\nabla b_p\times\partial_p u_k\right) - \partial_3^3 H_k\partial_3^3\left(\sum_{p=1}^{3}\nabla b_p\times\partial_p u_i\right) \\
&-\partial_3^3 H_i\sum_{l=1}^{3}\mathcal{C}_3^l\partial_3^l b\cdot\nabla\partial_3^{3-l}\omega_k - \partial_3^3 H_k\sum_{l=1}^{3}\mathcal{C}_3^l\partial_3^l b\cdot\nabla\partial_3^{3-l}\omega_i\Bigg)\cdot(b_j|\partial_2 u_j)\,dx.
\end{aligned}$$

By Lemma 2.1, it is easy to derive

$$\begin{aligned}
K_3 \lesssim{}& \|H\|_{H^3}^{\frac12}\|\partial_{\mathbf{1}} H\|_{H^3}^{\frac12}\big(\|u\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}} u\|_{H^4}^{\frac12}\|b\|_{H^3}^{\frac12}\|\partial_{\mathbf{2}} b\|_{H^3}^{\frac12} \\
&+\|b\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}} b\|_{H^4}^{\frac12}\|u\|_{H^3}^{\frac12}\|\partial_{\mathbf{2}} u\|_{H^3}^{\frac12}\big)\cdot\|(b|\partial_2 u)\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{H^1}^{\frac12}.
\end{aligned}$$

The last four parts of (4.9) are included in K_4:

$$\begin{aligned}
K_4 ={}& \int_{\mathbb{R}^3}\Big[-\partial_3^3 H_i\partial_3^3\Delta_h H_k - \partial_3^3 H_k\partial_3^3\Delta_h H_i\Big](b_j|\partial_2 u_j)dx \\
&+\int_{\mathbb{R}^3}\Big[-\partial_3^3 H_i\partial_3^3\partial_2\omega_k - \partial_3^3 H_k\partial_3^3\partial_2\omega_i\Big](b_j|\partial_2 u_j)dx \\
\lesssim{}& \|\nabla_h H\|_{H^3}^2\|(b|\partial_2 u)\|_{H^2} + \|H\|_{H^3}\|\nabla_h H\|_{H^3}\|\nabla_h(b|\partial_2 u)\|_{H^2} \\
&+\|\partial_2\partial_3^3 H\|_{L^2}\|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}\|(b|\partial_2 u)\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}(b|\partial_2 u)\|_{H^1}^{\frac12} \\
&+\|\partial_3^3 H\|_{L^2}^{\frac12}\|\partial_{\mathbf{2}}\partial_3^3 H\|_{L^2}^{\frac12}\|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_2(b|\partial_2 u)\|_{L^2}^{\frac12}\|\partial_{\mathbf{3}}\partial_2(b|\partial_2 u)\|_{L^2}^{\frac12}.
\end{aligned}$$

We have estimated all the terms corresponding to the right-hand side of (4.9) and thus obtained a suitable upper bound for J_{411} in (4.4). Integrating the upper bounds on K_1 through K_4 in time yields

$$\int_0^t |J_{411}|d\tau \lesssim \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t).$$

Together with the estimate (4.5) for J_{412}, we conclude that

$$\int_0^t |J_{41}|d\tau \lesssim \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t).$$

J_{42} has been estimated before in (4.3). Thus,

$$\int_0^t |J_4|d\tau \lesssim \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t).$$

We deal with the last term:

$$J_5 = \int_{\mathbb{R}^3} \left[\partial_3^3\omega_i\partial_3^3(H\cdot\nabla b_k - \partial_2 H_k) + \partial_3^3\omega_k\partial_3^3(H\cdot\nabla b_i - \partial_2 H_i)\right](b_j|\partial_2 u_j)\,dx.$$

By Lemma 2.1,

$$\begin{aligned} J_5 \lesssim\; & \|\omega\|_{H^3}^{\frac12}\|\partial_{\mathbf{1}}\omega\|_{H^3}^{\frac12}\|b\|_{H^4}^{\frac12}\|\partial_{\mathbf{1}}b\|_{H^4}^{\frac12}\|b\|_{H^3}^{\frac12}\|\partial_{\mathbf{2}}b\|_{H^3}^{\frac12}\|(b|\partial_2u)\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}(b|\partial_2u)\|_{H^1}^{\frac12} \\ & + \|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_3^3\partial_2H\|_{L^2}\|(b|\partial_2u)\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}(b|\partial_2u)\|_{H^1}^{\frac12}. \end{aligned}$$

Integrating in time yields

$$\int_0^t |J_5|\,d\tau \lesssim \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t).$$

Integrating (4.2) in time over $[0,t]$ and combining all the bounds for J_1 through J_5, we are led to the conclusion of Lemma 4.1. ◻

We are now ready to prove Proposition 1.1.

Proof of Proposition 1.1. Recall that the goal here is to bound the interaction terms $\mathcal{W}(t)$ defined by

$$\mathcal{W}^{ijk}(t) \triangleq \int_{\mathbb{R}^3} \partial_3^3\omega_i\partial_2u_j\partial_3^3\omega_k\,dx, \quad i,j,k\in\{1,2,3\}.$$

We replace ∂_2u_j by the equation of b_j in (1.2):

$$\begin{aligned} \mathcal{W}^{ijk}(t) &= \int_{\mathbb{R}^3}\partial_3^3\omega_i\Big[\partial_tb_j + u\cdot\nabla b_j - \Delta_hb_j - b\cdot\nabla u_j\Big]\partial_3^3\omega_k\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}^3}\partial_3^3\omega_ib_j\partial_3^3\omega_kdx - \int_{\mathbb{R}^3}b_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)dx \\ &\quad + \int_{\mathbb{R}^3}\partial_3^3\omega_iu\cdot\nabla b_j\partial_3^3\omega_kdx - \int_{\mathbb{R}^3}\partial_3^3\omega_ib\cdot\nabla u_j\partial_3^3\omega_kdx \\ &\quad - \int_{\mathbb{R}^3}\partial_3^3\omega_i\Delta_hb_j\partial_3^3\omega_kdx \\ &= \mathcal{W}_1^{ijk} + \mathcal{W}_2^{ijk} + \mathcal{W}_3^{ijk} + \mathcal{W}_4^{ijk} + \mathcal{W}_5^{ijk}. \end{aligned}$$

By Hölder's inequality,

$$\int_0^t W_1^{ijk}(\tau)\,d\tau \lesssim \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t).$$

$\mathcal{W}_2^{ijk}$ is an extremely difficult term. Fortunately, we have bounded it in Lemma 4.1:

$$\left|\int_0^t \mathcal{W}_2^{ijk}(\tau)d\tau\right| \lesssim \mathscr{E}^{\frac32}(0) + \mathscr{E}^{\frac32}(t) + \mathscr{E}^2(t).$$

By Lemma 2.1,

$$\begin{aligned} \mathcal{W}_3^{ijk} \lesssim\; & \|\partial_3^3\omega_i\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega_i\|_{L^2}^{\frac12}\|\partial_3^3\omega_k\|_{L^2}^{\frac12}\|\partial_{\mathbf{1}}\partial_3^3\omega_k\|_{L^2}^{\frac12} \\ & \cdot\|u\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}u\|_{H^1}^{\frac12}\|\nabla b_j\|_{H^1}^{\frac12}\|\partial_{\mathbf{2}}\nabla b_j\|_{H^1}^{\frac12} \end{aligned}$$

and

$$\begin{aligned}\mathcal{W}_4^{ijk} \lesssim{}& \|\partial_3^3\omega_i\|_{L^2}^{\frac12}\|\partial_{\boldsymbol{1}}\partial_3^3\omega_i\|_{L^2}^{\frac12}\|\partial_3^3\omega_k\|_{L^2}^{\frac12}\|\partial_{\boldsymbol{1}}\partial_3^3\omega_k\|_{L^2}^{\frac12}\\ &\cdot\|b\|_{H^1}^{\frac12}\|\partial_{\boldsymbol{2}}b\|_{H^1}^{\frac12}\|\nabla u_j\|_{H^1}^{\frac12}\|\partial_{\boldsymbol{2}}\nabla u_j\|_{H^1}^{\frac12}.\end{aligned}$$

Integrating in time yields

$$\int_0^t|\mathcal{W}_3^{ijk}(\tau)|d\tau,\ \int_0^t|\mathcal{W}_4^{ijk}(\tau)|d\tau\lesssim\mathscr{E}^2(t).$$

We divide $\mathcal{W}_5^{ijk}$ into two parts:

$$\begin{aligned}\mathcal{W}_5^{ijk}&=\int_{\mathbb{R}^3}\partial_1b_j\partial_1(\partial_3^3\omega_i\partial_3^3\omega_k)dx-\int_{\mathbb{R}^3}\partial_2^2b_j\partial_3^3\omega_i\partial_3^3\omega_k dx\\ &=\mathcal{W}_{51}^{ijk}+\mathcal{W}_{52}^{ijk}.\end{aligned}$$

The estimate for $\mathcal{W}_{51}^{ijk}$ is easy:

$$\mathcal{W}_{51}^{ijk}\lesssim\|\partial_1b\|_{H^2}\|\partial_1\omega\|_{H^3}\|\omega\|_{H^3}.$$

For $\mathcal{W}_{52}^{ijk}$, we replace ∂_2b_j via the equation of u_j in (1.2):

$$\mathcal{W}_{52}^{ijk}=-\int_{\mathbb{R}^3}\partial_2\big(\partial_tu_j+u\cdot\nabla u_j-\partial_1^2u_j-b\cdot\nabla b_j+\partial_jP\big)\partial_3^3\omega_i\partial_3^3\omega_k dx.$$

By Lemma 2.1 and Sobolev's inequality,

$$\begin{aligned}&-\int_{\mathbb{R}^3}\partial_2\big(u\cdot\nabla u_j-\partial_1^2u_j-b\cdot\nabla b_j+\nabla_jP\big)\partial_3^3\omega_i\partial_3^3\omega_k dx\\ &\quad\lesssim\|\partial_1u\|_{H^4}\|\partial_1\omega\|_{H^3}\|\omega\|_{H^3}+\|\partial_3^3\omega\|_{L^2}\|\partial_{\boldsymbol{1}}\partial_3^3\omega\|_{L^2}\big(\|u\|_{H^3}\|\partial_{\boldsymbol{2}}u\|_{H^3}+\|b\|_{H^3}\|\partial_{\boldsymbol{2}}b\|_{H^3}\big)\\ &\qquad+\|\partial_2\nabla P\|_{H^1}^{\frac12}\|\partial_{\boldsymbol{2}}\partial_2\nabla P\|_{H^1}^{\frac12}\|\partial_3^3\omega\|_{L^2}^{\frac32}\|\partial_{\boldsymbol{1}}\partial_3^3\omega\|_{L^2}^{\frac12}.\end{aligned}$$

Taking the divergence of the velocity equation in (1.2) yields a representation of the pressure P:

$$P=\sum_{i,j=1}^3(-\Delta)^{-1}\partial_i\partial_j(u_iu_j-b_ib_j).$$

Using the fact that Riesz operators are bounded on L^q for $1<q<\infty$ and the third inequality in Lemma 2.1, we have

$$\begin{aligned}&\|\partial_2P\|_{L^2}\lesssim\sum_{v=u,b}\|\partial_2(v\otimes v)\|_{L^2}\lesssim\sum_{v=u,b}\|\partial_2v\|_{L^2}^{\frac12}\|\partial_{\boldsymbol{1}}\partial_2v\|_{L^2}^{\frac12}\|v\|_{H^1}^{\frac12}\|\partial_{\boldsymbol{2}}v\|_{H^1}^{\frac12},\\ &\|\partial_2^2P\|_{L^2}\lesssim\sum_{v=u,b}\|\partial_2^2(v\otimes v)\|_{L^2}\lesssim\sum_{v=u,b}\Big(\|\partial_2v\|_{L^4}^2+\|\partial_2^2v\|_{L^2}^{\frac12}\|\partial_{\boldsymbol{1}}\partial_2^2v\|_{L^2}^{\frac12}\|v\|_{H^1}^{\frac12}\|\partial_{\boldsymbol{2}}v\|_{H^1}^{\frac12}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&-\int_{\mathbb{R}^3}\partial_2\big(u\cdot\nabla u_j-\partial_1^2u_j-b\cdot\nabla b_j+\nabla_jP\big)\partial_3^3\omega_i\partial_3^3\omega_k dx\\ &\quad\lesssim\|\partial_1u\|_{H^4}\|\partial_1\omega\|_{H^3}\|\omega\|_{H^3}+\|\partial_3^3\omega\|_{L^2}\|\partial_{\boldsymbol{1}}\partial_3^3\omega\|_{L^2}\big(\|u\|_{H^3}\|\partial_{\boldsymbol{2}}u\|_{H^3}+\|b\|_{H^3}\|\partial_{\boldsymbol{2}}b\|_{H^3}\big)\\ &\qquad+\sum_{v=u,b}\big(\|\partial_2v\|_{H^2}^{\frac12}\|\partial_{\boldsymbol{1}}\partial_2v\|_{H^2}^{\frac12}\|v\|_{H^3}^{\frac12}\|\partial_{\boldsymbol{2}}v\|_{H^3}^{\frac12}\big)^{\frac12}\\ &\qquad\quad\cdot\big(\|\partial_2v\|_{H^3}^2+\|\partial_2^2v\|_{H^2}^{\frac12}\|\partial_{\boldsymbol{1}}\partial_2^2v\|_{H^2}^{\frac12}\|v\|_{H^3}^{\frac12}\|\partial_{\boldsymbol{2}}v\|_{H^3}^{\frac12}\big)^{\frac12}\|\partial_3^3\omega\|_{L^2}^{\frac32}\|\partial_{\boldsymbol{1}}\partial_3^3\omega\|_{L^2}^{\frac12}.\end{aligned}$$

This implies that we only have one term left in the estimate for W_{52}^{ijk}, namely,

$$-\int_{\mathbb{R}^3} \partial_2\partial_t u_j \, \partial_3^3\omega_i\partial_3^3\omega_k dx.$$

We shift the time derivative via the equation

$$-\int_{\mathbb{R}^3} \partial_2\partial_t u_j\partial_3^3\omega_i\partial_3^3\omega_k dx = -\frac{d}{dt}\int_{\mathbb{R}^3} \partial_2 u_j\partial_3^3\omega_i\partial_3^3\omega_k dx + \int_{\mathbb{R}^3} \partial_2 u_j\partial_t(\partial_3^3\omega_i\partial_3^3\omega_k)dx.$$

The last term above is a difficult term and is bounded via Lemma 4.1. Integrating in time and invoking the upper bounds, we find

$$\int_0^t \left|\mathcal{W}_5^{ijk}(\tau)\right| d\tau \lesssim \mathscr{E}^{\frac{3}{2}}(0) + \mathscr{E}^{\frac{3}{2}}(t) + \mathscr{E}^2(t).$$

This completes the proof of Proposition 1.1. □

5. Proof of Theorem 1.1.

This section completes the proof of Theorem 1.1, which is achieved by applying the bootstrapping argument to the energy inequalities obtained in the previous sections.

Proof of Theorem 1.1. As aforementioned in the introduction, the local well-posedness of (1.2) in H^4 follows from a standard procedure (see, e.g., [37]), and our attention is exclusively focused on the global bound of H^4-norms. This is accomplished by the bootstrapping argument. The key components are the two energy inequalities established previously in sections 2 and 3,

$$\mathscr{E}_2(t) \le C_1\,\mathscr{E}(0) + C_2\,\mathscr{E}_1(t) + C_3\,\mathscr{E}_1^{\frac{3}{2}}(t) + C_4\,\mathscr{E}_2^{\frac{3}{2}}(t), \tag{5.1}$$

$$\mathscr{E}_1(t) \le C_5\,\mathscr{E}(0) + C_6\,\mathscr{E}^{\frac{3}{2}}(0) + C_7\,\mathscr{E}^{\frac{3}{2}}(t) + C_8\,\mathscr{E}^2(t), \tag{5.2}$$

where $\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2$. Adding (5.2) to $1/(2C_2)$ of (5.1) yields, for a constant $C_0 > 0$ and for any $t > 0$,

$$\mathscr{E}(t) \le C_0\,\mathscr{E}(0) + C_0\,\mathscr{E}^{\frac{3}{2}}(0) + C_0\,\mathscr{E}^{\frac{3}{2}}(t) + C_0\,\mathscr{E}^2(t). \tag{5.3}$$

Without loss of generality, we assume $C_0 \ge 1$. Applying a bootstrapping argument to (5.3) then implies that, if $\|(u_0,b_0)\|_{H^4}$ is sufficiently small, say,

$$\mathscr{E}(0) \le \frac{1}{128C_0^3} \quad \text{or} \quad \|(u_0,b_0)\|_{H^4} \le \epsilon := \frac{1}{\sqrt{128C_0^3}}, \tag{5.4}$$

then, for any $t > 0$,

$$\mathscr{E}(t) \le \frac{1}{32C_0^2}, \quad \text{especially} \quad \|(u(t),b(t))\|_{H^4} \le 2\sqrt{C_0}\,\epsilon.$$

In fact, if we make the ansatz that

$$\mathscr{E}(t) \le \frac{1}{16C_0^2} \tag{5.5}$$

and insert (5.5) in (5.3), we obtain

$$\mathscr{E}(t) \le C_0\,\mathscr{E}(0) + C_0\,\mathscr{E}^{\frac{3}{2}}(0) + \frac{1}{2}\mathscr{E}(t),$$

which, together with (5.4), implies

$$\mathscr{E}(t) \le \frac{1}{32C_0^2}. \tag{5.6}$$

The bootstrapping argument then implies that (5.6) actually holds for all $t > 0$. This completes the proof of Theorem 1.1. ∎

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