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# The Global Well-Posedness and Decay Estimates for the 3D Incompressible MHD Equations With Vertical Dissipation in a Strip

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The three-dimensional incompressible magnetohydrodynamic (MHD) system with only vertical dissipation arises in the study of reconnecting plasmas. When the spatial domain is the whole space $\mathbb{R}^3$, the small data global well-posedness remains an extremely challenging open problem. The one-directional dissipation is simply not sufficient to control the nonlinearity in $\mathbb{R}^3$. This paper solves this open problem when the spatial domain is the strip $\Omega := \mathbb{R}^2 \times [0,1]$ with Dirichlet boundary conditions. By invoking suitable Poincaré type inequalities and designing a multi-step scheme to separate the estimates of the horizontal and the vertical derivatives, we are able to establish the global well-posedness in the Sobolev setting $H^3$ as long as the initial horizontal derivatives are small. We impose no smallness condition on the vertical derivatives of the initial data. Furthermore, the $H^3$-norm of the solution is shown to decay exponentially in time. This exponential decay is surprising for a system with no horizontal dissipation. This large-time behavior reflects the smoothing and stabilizing phenomenon due to the interaction within the MHD system and with the boundary.

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## 1 Introduction

This paper focuses on the following 3D incompressible magnetohydrodynamic (MHD) system with only vertical dissipation in a strip domain $\Omega = \mathbb{R}^2 \times [0,1]$,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\,\partial_{33}u + (b\cdot\nabla)b, \qquad x\in\emptyset,\ t>0,\\ \partial_t b + (u\cdot\nabla)b = \eta\,\partial_{33}b + (b\cdot\nabla)u,\\ \nabla\cdot u = \nabla\cdot b = 0,\\ u(x,0)=u_0(x), \quad b(x,0)=b_0(x) \end{cases} \tag{1.1}$$

supplemented with the Dirichlet boundary condition

$$u|_{\partial\Omega}=0, \quad b|_{\partial\Omega}=0,\ t>0.$$

Here $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $b = (b_1(x,t), b_2(x,t), b_3(x,t))$ denote the fluid velocity and the magnetic field, respectively, $p(x,t)$ the total pressure, and the parameters $\nu > 0$ and $\eta > 0$ represent the viscosity and resistivity, respectively. The MHD system is a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. They govern the motion of electrically conducting fluid such as plasmas, liquid metals, and electrolytes, and have a very wide range of applications in astrophysics, geophysics, cosmology, and engineering (see, e.g., [5, 7, 17, 41]). The MHD system (1.1) focused here is relevant in the modeling of reconnecting plasmas (see, e.g., [13, 14]).

The goal of this paper is twofold: first to solve the global well-posedness problem, second, to determine the precise large-time behavior of the solutions. The issues put forward for study here are not trivial and can not be dealt with via existing approaches. There are three immediate difficulties. The first is that the dissipation in only one direction is not sufficient to control the nonlinearity. Extra regularizing properties are needed in order to obtain time-integrable upper bounds for the nonlinear terms. In the case of whole space $\mathbb{R}^3$, exactly due to this difficulty, the small-data global well-posedness on (1.1) remains a challenging open problem. Clearly, we need to take advantage of the domain $\Omega$ and the associated boundary condition in order to solve the well-posedness problem focused here.

The second difficulty is due to the presence of the boundary. In the process of estimating the Sobolev norms of the solutions, we can no longer integrate by parts freely as in the whole space case. This forces us to design a more delicate scheme to avoid the boundary terms. Observing that the horizontal derivatives of the solution are all zero on the boundary due to the boundary conditions, we need to distinguish the estimates of the horizontal derivatives from those of the vertical derivatives. In addition, we need to estimate the time derivatives in order to control the vertical derivatives. This explains why the estimates on the Sobolev norms of the solutions are much more involved than those in the whole space case.

The third difficulty arises in the study of the large-time behavior. Powerful methods have been created to determine the large-time behavior of fully dissipative systems of partial differential equations (PDEs). Schonbek's Fourier splitting scheme has worked very well when the Navier–Stokes, the Boussinesq, or the MHD equations involve full dissipation (see, e.g., [25, 44]). However, these methods can not be extended to partially dissipated PDE systems. In fact, no existing method can be adapted to deal with the MHD system with dissipation in only one direction. This paper intends to develop new approaches that are capable of extracting the large-time behavior of anisotropic PDE systems. This paper is able to resolve all three difficulties described here and successfully establish the desired well-posedness and large-time behavior.

To give a precise account of our main result, we introduce the following notations and norms,

$$v_h = (v_1, v_2), \quad \nabla_h v = (\partial_1 v, \partial_2 v), \quad \Delta_h v = \partial_1^2 v + \partial_2^2 v,$$

$$\|f\|_{H^{s,0}(\Omega)}^2 = \sum_{i=1}^{2} \sum_{0\le|\alpha|\le s} \|\partial_i^\alpha f\|_{L^2(\Omega)}^2,$$

$$\|(f,g)\|_{H^s}^2 = \|f\|_{H^s}^2 + \|g\|_{H^s}^2, \quad \|(f,g)\|_{H^{s,0}}^2 = \|f\|_{H^{s,0}}^2 + \|g\|_{H^{s,0}}^2.$$

Our main result can then be stated as follows.

**Theorem 1.1.** Assume that the initial data satisfies $(u_0, b_0) \in H^3(\Omega)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$, and the zero boundary conditions on $\partial\Omega$. Then there exists $\delta > 0$ such that, if

$$\|u_0\|_{H^{3,0}(\Omega)} + \|b_0\|_{H^{3,0}(\Omega)} \le \delta,$$

then the 3D MHD system (1.1) admits a unique global solution $(u, b)$ satisfying

$$\|(u(t), b(t))\|_{H^3(\Omega)}^2 + \|(\partial_t u(t), \partial_t b(t))\|_{H^1(\Omega)}^2 + \|\nabla p(t)\|_{H^1(\Omega)}^2 + 2\nu \int_0^t \|\partial_3 u(\tau)\|_{H^{3,0}(\Omega)}^2 \, d\tau$$
$$+ 2\eta \int_0^t \|\partial_3 b(\tau)\|_{H^{3,0}(\Omega)}^2 d\tau \le C\Big(\delta^2 + \delta^4 + \|(\partial_3 u_0, \partial_3 b_0)\|_{H^1}^4 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{H^1}^2\Big)$$

for any $t > 0$ and some uniform constant $C$. In addition, the following decay estimate holds:

$$\|(u(t), b(t))\|_{H^3(\Omega)} + \|(\partial_t u(t), \partial_t b(t))\|_{H^1(\Omega)} + \|\nabla p(t)\|_{H^1(\Omega)} \leq Ce^{-C^* t} \tag{1.2}$$

for some constants $C > 0$ and $C^* > 0$.

We make several remarks on Theorem 1.1.

**Remark 1.2.** The smallness condition on the initial data is only imposed on the $L^2$-norms of $(u_0, b_0)$ and its horizontal derivatives. There is no requirement on the vertical derivatives. Therefore, the $H^3$-norm of $(u_0, b_0)$ is not necessarily small. In this sense, our result is actually a global well-posedness without smallness assumption on the full initial $H^3$-norm.

**Remark 1.3.** The exponential decay estimate for $\|(u(t), b(t))\|_{H^3}$ in (1.2) is surprising if we take into account of the fact that the MHD system concerned here has no horizontal dissipation and the horizontal variables are in the whole space $\mathbb{R}^2$. This remarkable large-time behavior does not directly come from the dissipation in the system, but rather is a consequence of the smoothing and stabilizing effect of the interactions within the MHD system and with the boundary.

**Remark 1.4.** A special consequence of Theorem 1.1 is the global well-posedness of the 3D incompressible Navier–Stokes equations in a strip domain,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \partial_{33} u, \quad x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.3}$$

when the horizontal derivatives of $u_0$ is sufficiently small. More precisely, if

$$\|u_0\|_{H^{3,0}(\Omega)} \leq \delta$$

for sufficiently small $\delta > 0$, then (1.3) has a unique global solution $u \in L^\infty(0, \infty; H^3(\Omega))$. In addition, $(u, p)$ decays exponentially in the sense that, for two positive constants $C, C^*$

and for all $t > 0$,

$$\|u(t)\|_{H^3(\Omega)} + \|\partial_t u(t)\|_{H^1(\Omega)} + \|\nabla p(t)\|_{H^1(\Omega)} \le Ce^{-C^* t}.$$

We remark that an important work by Paicu and Zhang [39] has investigated the small data global well-posedness of (1.3) in the anisotropic Besov setting.

To place our result in a suitable context of existing research, we describe some of the closely related work on the incompressible MHD equations. Due to their physical applications and mathematical importance, the global well-posedness, stability, and large-time behavior problems on the MHD equations have recently attracted considerable interests from the community of mathematical fluid mechanics. Many recent efforts are devoted to various partially or fractionally dissipated MHD systems. Since classical approaches designed for systems with full dissipation no longer work, new techniques and methods have recently been developed to deal with anisotropic MHD equations. Significant progress has been made. Existence and regularity results for the 2D MHD equations with various partial or fractional dissipation has been established (see, e.g., [9–11, 19, 20, 31–33, 46, 54, 55]). Local and global well-posedness on the 3D MHD equations with standard dissipation or various form of hyperdissipation has also been obtained (see, e.g., [12, 21, 28, 34, 35, 45, 48–50, 53, 56, 57]). The study on the well-posedness and stability problem on the MHD equations near the trivial solution or a background magnetic field has recently gained a lot of momentum. There are substantial developments (see, e.g., [3, 6, 8, 18, 22, 26, 27, 29, 30, 36, 37, 40, 42, 43, 47, 51, 52, 58–60]). Especially, [3, 6, 22, 26, 36, 52] reveal and rigorously confirm the stabilizing phenomenon observed in physical experiments on MHD turbulence (see, e.g., [1, 2, 15–17, 23, 24]). We remark that previous work on the 3D anisotropic MHD equations requires either the velocity equation or the equation of the magnetic field has dissipation in at least two directions. Consequently, none of the previous approaches can be applied directly to solve the problems concerned here.

We explain the main idea in the proof of Theorem 1.1. Naturally the proof is divided into two main parts with the first devoted to the global well-posedness and the second to the decay estimate. The center piece of the global well-posedness is the global bound on $\|(u, b)\|_{H^3}$. Due to the presence of the boundary, we can no longer integrate by parts freely as in the whole space case. Our observation is that the horizontal derivatives of the solution are all zero on the boundary due to the boundary conditions on $u$ and $b$. On the contrary, the boundary-values of the vertical derivatives are unknown. To accommodate this observation, we design a multi-step scheme to estimate $\|(u,b)\|_{H^3}$. The first step focuses on the norm of the horizontal derivatives, namely $\|(u,b)\|_{H^{3,0}}$. With no boundary terms generated in the process, our attention focuses on how to control the nonlinearity by the vertical dissipation. To make up for the lack of dissipation in two directions, we make use of the boundary conditions to derive strong versions of Poincaré type inequalities such as

$$\|f\|_{L^2(\Omega)} \le C\|\partial_3 f\|_{L^2(\Omega)},$$
$$\|f\|_{L^\infty(\Omega)} \le C\|\partial_3 f\|_{H^{2,0}(\Omega)},$$

which are valid for any functions $f$ with zero boundary conditions. More information can be found in Section 2. In addition, we use various anisotropic upper bounds for triple products generated from the nonlinearity. After a long process of estimating many terms, we are able to obtain the following energy inequality,

$$\|(u,b)\|_{H^{3,0}}^2 + \min\{\nu,\eta\}\int_0^t \left(\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2\right) d\tau$$
$$\le C\|(u_0,b_0)\|_{H^{3,0}}^2 + C\int_0^t \|(u,b)(\tau)\|_{H^{3,0}}\left(\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2\right) d\tau.$$

Remarkably this inequality is self-contained and involves no vertical derivatives. An application of the bootstrapping argument then yields a global bound on $\|(u,b)\|_{H^{3,0}}$ as well as on $\int_0^t \|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2\, d\tau$ when the initial norm $\|(u_0,b_0)\|_{H^{3,0}}$ is sufficiently small.

However, the lack of boundary conditions on the vertical derivatives prevents us from estimating the norms of the vertical derivatives directly. Our strategy is to rewrite the equation of $u$ in (1.1) as

$$\begin{cases} -\Delta u + \nabla p = f, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\emptyset,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0. \end{cases} \tag{1.4}$$

with

$$f := -\partial_t u - (u\cdot\nabla)u + (b\cdot\nabla)b - \Delta_h u,$$

and the equation of $b$ in (1.1) as

$$\begin{cases} -\Delta b = g, & x \in \Omega,\ t > 0, \\ b(x,t) = 0, & x \in \partial\emptyset,\ t > 0, \\ \nabla \cdot b = 0, & x \in \Omega,\ t > 0 \end{cases} \tag{1.5}$$

with

$$g := -\partial_t b - (u \cdot \nabla) b + (b \cdot \nabla) u - \Delta_h b.$$

The regularity theory on the Stokes system in (1.4) and the Poisson equation in (1.5) then converts the estimates of the $H^3$-norm of $(u, b)$ into the estimates of the $H^1$-norm of $f$ and $g$. In particular, we need to bound $\|(\partial_t u, \partial_t b)\|_{H^1}$. This is accomplished in the second step. Naturally, this step is divided into the estimates of $\|(\partial_t u, \partial_t b)\|_{L^2}$ and of $\|(\nabla\partial_t u, \nabla\partial_t b)\|_{L^2}$. Moreover, due to the lack of boundary conditions for the vertical derivatives, we need to further write

$$\|(\nabla\partial_t u, \nabla\partial_t b)\|_{L^2}^2 = \|(\nabla_h\partial_t u, \nabla_h\partial_t b)\|_{L^2}^2 + \|(\partial_3\partial_t u, \partial_3\partial_t b)\|_{L^2}^2$$

and deal with the horizontal derivatives and the vertical derivatives accordingly. We are able to show that

$$\begin{aligned} &\|u_t\|_{L^2} + \|b_t\|_{L^2} \le Ce^{C\int_0^t \|(\partial_3 u, \partial_3 b)(\tau)\|_{H^{3,0}}^2 d\tau} \left(\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}\right), \\ &\|\nabla u_t\|_{L^2} + \|\nabla b_t\|_{L^2} \le Ce^{C\int_0^t \left(1+\|(\nabla u, \nabla b)(\tau)\|_{L^2}^2\right)\|(\partial_3 u, \partial_3 b)(\tau)\|_{H^{3,0}}^2 d\tau} \\ &\qquad\qquad \times \left(\|(u_0, b_0)\|_{H^2}^2 + \|(\nabla\partial_3^2 u_0, \nabla\partial_3^2 b_0)\|_{L^2}\right). \end{aligned}$$

The final step is to invoke the regularity theory on the Stokes system and the Poisson equation to establish the desired global bound on $\|(u, b)\|_{H^3}$.

The exponential decay estimate (1.2) in Theorem 1.1 is established through three stages. The first stage proves the exponential decay rate for $\|(u, b)\|_{H^{3,0}} + \|(\partial_3 u, \partial_3 b)\|_{L^2}$. This is achieved by first deriving an equation for $\partial_t(\nu\|\partial_3 u\|_{L^2}^2 + \eta\|\partial_3 b\|_{L^2}^2)$ and then combining with the equation for $\partial_t\|(u, b)\|_{H^{3,0}}$ previously obtained in the well-posedness part. This process leads to a differential inequality of the form

$$\partial_t X(t) + CX(t) \le 0$$

for a constant $C > 0$ and $X(t) = \|(u,b)\|_{H^{3,0}} + \|(\partial_3 u, \partial_3 b)\|_{L^2}$. The second stage focuses on the exponential decay rate for $\|(\partial_t u, \partial_t b)\|_{H^1}$,

$$\|(\partial_t u, \partial_t b)(t)\|_{H^1} \leq C e^{-Ct}, \tag{1.6}$$

where $C > 0$ is a constant. (1.6) is verified by first deriving a refined energy inequality for

$$\|(\partial_t u, \partial_t b)\|_{H^{1,0}}^2 + \nu \|\partial_3 \partial_t u\|_{L^2}^2 + \eta \|\partial_3 \partial_t b\|_{L^2}^2.$$

The precise inequality and its proof are provided in Proposition 4.2. The final stage invokes the regularity estimates of the Stokes system (1.4) and the Poisson equation (1.5), and combines the exponential rates from the first two stages.

The rest of this paper is divided into three sections. Section 2 presents three tool lemmas to be used in the proof of Theorem 1.1. The first contains three Poincaré-type inequalities, the second provides several anisotropic upper bounds for triple products, whereas the third states the the existence and regularity result on a Stokes system with no-slip boundary condition. Section 3 is devoted to the proof of the global well-posedness part of Theorem 1.1. It is further divided into four subsections. Section 4 proves the exponential decay estimate of Theorem 1.1. It first derives two main propositions and then use them to establish the desired decay estimates.

## 2 Preliminary

This section prepares three tool lemmas to be used in the proof of Theorem 1.1. The first lemma provides several Poincaré-type inequalities, which allow us to bound the $L^2$-, $L^\infty$- and $L^4$-norms of a function $f$ defined on $\Omega$ in terms of suitable norms of $\partial_3 f$. This is one of the reasons that we can control the nonlinearity of the MHD system in terms of the vertical dissipation. They play a crucial role in achieving the time-integrable upper bounds for the nonlinear terms.

The second lemma presents several anisotropic upper bounds for triple products. Nonlinear terms in the MHD system emerge as triple products in the estimates of the norms on the solutions, and this lemma can bound such products by selectively placing partial derivatives on the components of the products. This helps maximally make use of the anisotropic dissipation. These type of inequalities have proven to be extremely important in the study of the 2D anisotropic PDEs (see, e.g., [10]) as well as 3D anisotropic PDEs (see, e.g., [52]).

The third lemma states the existence and regularity result on a Stokes system defined on $\Omega$ with no-slip boundary condition. It will be used to estimate the vertical derivatives of the solutions such as $\|\partial_3 u\|_{H^2(\Omega)}$.

We now state and prove the first lemma that contains several Poincaré-type inequalities. Standard Poincaré inequalities require the gradient, but what we need here is mainly the $x_3$-directional derivative.

**Lemma 2.1.** Let $\Omega = \mathbb{R}^2 \times [0,1]$. Assume $f|_{\partial\Omega} = 0$, $f \in H^{1,0}(\Omega)$ and $\partial_3 f \in H^{2,0}(\Omega)$. Then for some constant $C > 0$, we have

$$\|f\|_{L^2(\Omega)} \le C\|\partial_3 f\|_{L^2(\Omega)}, \tag{2.1}$$

$$\|f\|_{L^\infty(\Omega)} \le C\|\partial_3 f\|_{H^{2,0}(\Omega)}, \tag{2.2}$$

$$\|f\|_{L^4(\Omega)} \le C\|f\|_{L^2}^{\frac{1}{4}}\|\nabla_h f\|_{L^2}^{\frac{1}{2}}\|\partial_3 f\|_{L^2}^{\frac{1}{4}} \le C\|\nabla_h f\|_{L^2}^{\frac{1}{2}}\|\partial_3 f\|_{L^2}^{\frac{1}{2}}. \tag{2.3}$$

**Proof of Lemma 2.1** According to the one-dimensional Poincaré inequality,

$$\|f\|_{L^2_{x_3}} \le C\|\partial_3 f\|_{L^2_{x_3}}.$$

Squaring each side of the inequality above and integrating over $(x_1, x_2) \in \mathbb{R}^2$ yield

$$\|f\|_{L^2(\Omega)} \le C\|\partial_3 f\|_{L^2(\Omega)}.$$

Due to $f|_{\partial\Omega} = 0$, by Hölder's inequality and Poincaré's inequality, we have

$$\|f\|_{L^\infty_{x_3}} \le \sqrt{2}\,\|f\|_{L^2_{x_3}}^{\frac{1}{2}}\|\partial_3 f\|_{L^2_{x_3}}^{\frac{1}{2}} \le C\|\partial_3 f\|_{L^2_{x_3}}.$$

By Minkowski's inequality and the Sobolev imbedding inequality,

$$\begin{aligned}\|f\|_{L^\infty} &= \Big\| \|f\|_{L^\infty_{x_3}} \Big\|_{L^\infty_{x_1,x_2}} \le C\Big\| \|\partial_3 f\|_{L^\infty_{x_1,x_2}} \Big\|_{L^2_{x_3}} \\ &\le C\Big\| \|\partial_3 f\|_{H^2_{x_1,x_2}} \Big\|_{L^2_{x_3}} = C\|\partial_3 f\|_{H^{2,0}}.\end{aligned}$$

Similarly,

$$
\begin{aligned}
\|f\|_{L^4} = \Big\| \|f\|_{L^4_{x_3}} \Big\|_{L^4_{x_1,x_2}} &\le C \Big\| \|f\|^{\frac{3}{4}}_{L^2_{x_3}} \|\partial_3 f\|^{\frac{1}{4}}_{L^2_{x_3}} \Big\|_{L^4_{x_1,x_2}} \\
&\le C \Big\| \|f\|_{L^6_{x_1,x_2}} \Big\|^{\frac{3}{4}}_{L^2_{x_3}} \|\partial_3 f\|^{\frac{1}{4}}_{L^2} \\
&\le C \Big\| \|f\|^{\frac{1}{3}}_{L^2_{x_1,x_2}} \|\nabla_h f\|^{\frac{2}{3}}_{L^2_{x_1,x_2}} \Big\|^{\frac{3}{4}}_{L^2_{x_3}} \|\partial_3 f\|^{\frac{1}{4}}_{L^2} \\
&\le C \|f\|^{\frac{1}{4}}_{L^2} \|\nabla_h f\|^{\frac{1}{2}}_{L^2} \|\partial_3 f\|^{\frac{1}{4}}_{L^2} \le C \|\nabla_h f\|^{\frac{1}{2}}_{L^2} \|\partial_3 f\|^{\frac{1}{2}}_{L^2}.
\end{aligned}
$$

This completes the proof of of Lemma 2.1. ■

The second lemma presents several anisotropic inequalities for triple products, which play a crucial role in establishing the global bound for $\|(u,b)\|_{H^{3,0}(\Omega)}$ and for the decay estimates. This lemma can be shown by means of the proof in [52] together with Poincaré inequality $\|f\|_{L^\infty_{x_3}} \le C\|\partial_3 f\|_{L^2_{x_3}}$ for $f|_{\partial\Omega} = 0$.

**Lemma 2.2.** Let $\Omega = \mathbb{R}^2 \times [0,1]$. Assume $f|_{\partial\Omega} = 0$. Then,

$$
\int |fgh|\,dx \le C\|\partial_3 f\|_{L^2(\Omega)} \|g\|^{\frac{1}{2}}_{L^2(\Omega)} \|\partial_1 g\|^{\frac{1}{2}}_{L^2(\Omega)} \|h\|^{\frac{1}{2}}_{L^2(\Omega)} \|\partial_2 h\|^{\frac{1}{2}}_{L^2(\Omega)}, \tag{2.4}
$$

$$
\begin{aligned}
\int |fgh|\,dx &\le C\|f\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_3 f\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_2 f\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_2\partial_3 f\|^{\frac{1}{4}}_{L^2(\Omega)} \|g\|^{\frac{1}{2}}_{L^2(\Omega)} \|\partial_1 g\|^{\frac{1}{2}}_{L^2(\Omega)} \|h\|_{L^2(\Omega)} \\
&\le C\|\partial_3 f\|^{\frac{1}{2}}_{L^2(\Omega)} \|\partial_2 f\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_2\partial_3 f\|^{\frac{1}{4}}_{L^2(\Omega)} \|g\|^{\frac{1}{2}}_{L^2(\Omega)} \|\partial_1 g\|^{\frac{1}{2}}_{L^2(\Omega)} \|h\|_{L^2(\Omega)}.
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
\int |fgh|\,dx &\le C\|f\|^{\frac{1}{2}}_{L^2(\Omega)} \|\partial_3 f\|^{\frac{1}{2}}_{L^2(\Omega)} \|g\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_1 g\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_1\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)} \|h\|_{L^2(\Omega)} \\
&\le C\|\partial_3 f\|_{L^2(\Omega)} \|g\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_1 g\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)} \|\partial_1\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)} \|h\|_{L^2(\Omega)}.
\end{aligned} \tag{2.6}
$$

The last lemma of this section states the existence and regularity result on a Stokes system with no-slip boundary condition. This lemma is taken from Beirao da Veiga [4].

**Lemma 2.3** (Stokes estimates). Let $\Omega = \mathbb{R}^2 \times [0,1]$ be the strip domain. Let $f \in H^k(\Omega)$ with $k \ge 0$ being an integer. Assume $v \in H^1(\Omega)$ is the weak solution of the Stokes equations

$$\begin{cases} -\triangle v + \nabla P = f, & \text{in } \Omega, \\ \nabla \cdot v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Then (2.7) has a unique strong solution $(v, P) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and the following estimate

$$\|\nabla v\|_{H^{k+1}(\Omega)} + \|\nabla P\|_{H^k(\Omega)} \le C\|f\|_{H^k(\Omega)}, \tag{2.8}$$

holds for some positive constant C.

## 3 The global well-posedness

This section proves the global well-posedness part of Theorem 1.1. Since the local well-posedness can be shown via a standard procedure (see, e.g., [38]), our attention will be focused on the global bounds. As aforementioned in the introduction, we need to distinguish the estimates of the horizontal derivatives from those of the vertical ones. In addition, we also need to estimate the time derivatives in order to achieve suitable bounds on the vertical derivatives. The whole process involves the estimates of many terms and is very lengthy. For the sake of clarity, we split the proof into four parts. The first focuses on the estimates of $\|(u, b)\|_{H^{3,0}}$, the second bounds $\|(u_t, b_t)\|_{H^1}$, whereas the third presents the estimates on $\|(u, b)\|_{H^3}$. The last part assembles the energy inequalities from the first three parts, establishes the desired global bounds on $(u, b)$ and thus finishes the proof on the global well-posedness. Naturally, we divide the rest of this section into four subsections.

### 3.1 Estimates for $\|(u, b)\|_{H^{3,0}}$

This subsection estimates the horizontal derivatives of the solution. We use the crucial fact that, due to the boundary conditions on $u$ and $b$, the horizontal derivatives of $u$ and $b$ are also zero on $\partial\Omega$, namely

$$\partial_i^k u|_{\partial\Omega} = \partial_i^k b|_{\partial\Omega} = 0$$

for any $i = 1, 2$ and $k = 1, 2, 3$.

**Proposition 3.1.** Assume $(u_0, b_0) \in H^{3,0}$ and let $(u, b)$ be the corresponding solution to (1.1). Then, $(u, b)$ satisfies

$$\|(u,b)\|_{H^{3,0}}^2 + \min\{\nu,\eta\}\int_0^t \left(\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2\right) d\tau$$
$$\leq C\|(u_0,b_0)\|_{H^{3,0}}^2 + C\int_0^t \|(u,b)(\tau)\|_{H^{3,0}}\left(\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2\right) d\tau. \tag{3.1}$$

**Proof of Proposition 3.1** Taking the $L^2$-inner product of (1.1) with $(u, b)$, integrating by parts and applying $\nabla\cdot u = \nabla\cdot b = 0$ and the boundary conditions, we find

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \left(\nu\|\partial_3 u\|_2^2 + \eta\|\partial_3 b\|_2^2\right) = 0. \tag{3.2}$$

Since the norm $\|(u,b)\|_{H^{3,0}}$ is equivalent to $\|(u,b)\|_{L^2} + \|(u,b)\|_{\dot{H}^{3,0}}$ in $\Omega$, it suffices to establish the estimate on $\|(u,b)\|_{\dot{H}^{3,0}}$. Applying $\partial_i^3$ $(i = 1, 2)$ to (1.1) and taking $L^2$-inner product of the resulting equations with $\partial_i^3 u$ and $\partial_i^3 b$, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\left(\|\partial_i^3 u\|_{L^2}^2 + \|\partial_i^3 b\|_{L^2}^2\right) + \sum_{i=1}^{2}\left(\nu\|\partial_i^3\partial_3 u\|_{L^2}^2 + \eta\|\partial_i^3\partial_3 b\|_{L^2}^2\right)$$
$$:= I_1 + I_2 + I_3 + I_4, \tag{3.3}$$

where

$$I_1 = -\sum_{i=1}^{2}\int \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\, dx,$$
$$I_2 = \sum_{i=1}^{2}\int [\partial_i^3(b\cdot\nabla b) - b\cdot\nabla\partial_i^3 b]\cdot\partial_i^3 u\, dx,$$
$$I_3 = -\sum_{i=1}^{2}\int \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\, dx,$$
$$I_4 = \sum_{i=1}^{2}\int [\partial_i^3(b\cdot\nabla u) - b\cdot\nabla\partial_i^3 u]\cdot\partial_i^3 b\, dx.$$

Here, we have used

$$\int \partial_i^3 u\cdot\partial_i^3\nabla p\, dx = 0 \quad \text{and} \quad \int b\cdot\nabla(\partial_i^3 u\cdot\partial_i^3 b)\, dx = 0$$

by integration by parts, $\nabla\cdot u = \nabla\cdot b = 0$ and the boundary conditions $\partial_i^3 u|_{\partial\Omega} = \partial_i^3 b|_{\partial\Omega} = 0$. To bound $I_1$, we write $u\cdot\nabla u = u_h\cdot\nabla_h u + u_3\partial_3 u$ to decompose $I_1$ into two parts,

$$\begin{aligned} I_1 &= -\sum_{i=1}^{2}\int \partial_i^3(u_h\cdot\nabla_h u)\cdot\partial_i^3 u\,dx - \sum_{i=1}^{2}\int \partial_i^3(u_3\,\partial_3 u)\cdot\partial_i^3 u\,dx \\ &:= I_{1,1} + I_{1,2}. \end{aligned}$$

By Leibniz's formula, we further split $I_{1,1}$ into three terms according to the order $k$ of the derivative.

$$I_{1,1} = -\sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int \partial_i^k u_h\cdot\partial_i^{3-k}\nabla_h u\cdot\partial_i^3 u\,dx := I_{1,1,1} + I_{1,1,2} + I_{1,1,3}.$$

By Hölder's inequality, $\nabla_h u|_{\partial\Omega} = \nabla_h^3 u|_{\partial\Omega} = 0$, and (2.1) and (2.2),

$$\begin{aligned} I_{1,1,1} + I_{1,1,3} &= -3\sum_{i=1}^{2}\int \partial_i u_h\cdot\partial_i^2\nabla_h u\cdot\partial_i^3 u\,dx - \sum_{i=1}^{2}\int \partial_i^3 u_h\cdot\nabla_h u\cdot\partial_i^3 u\,dx \\ &\le C\sum_{i=1}^{2}\|\nabla_h u_h\|_{L^\infty}\|\nabla_h^3 u\|_{L^2}\|\partial_i^3 u\|_{L^2} \\ &\le C\|\partial_3\nabla_h u_h\|_{H^{2,0}}\|\partial_3 u\|_{H^{3,0}}\|u\|_{H^{3,0}} \\ &\le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2. \end{aligned} \tag{3.4}$$

Applying the anisotropic inequality (2.4) and the Poincaré-type inequality (2.1), we have

$$\begin{aligned} I_{1,1,2} &= -3\sum_{i=1}^{2}\int \partial_i^2 u_h\cdot\nabla_h\partial_i u\cdot\partial_i^3 u dx \\ &\le C\sum_{i=1}^{2}\|\partial_i^2 u\|_{L^2}^{\frac12}\|\partial_i^2\partial_1 u\|_{L^2}^{\frac12}\|\partial_i\nabla_h u\|_{L^2}^{\frac12}\|\partial_2\partial_i\nabla_h u\|_{L^2}^{\frac12}\|\partial_i^3\partial_3 u\|_{L^2} \\ &\le C\|\nabla_h^2 u\|_{L^2}\|\partial_3\nabla_h^3 u\|_{L^2}^2. \end{aligned} \tag{3.5}$$

For $I_{1,2}$, we split it into two parts,

$$I_{1,2} = -\sum_{i=1}^{2}\sum_{k=1}^{2} C_3^k \int \partial_i^k u_3\, \partial_i^{3-k}\partial_3 u \cdot \partial_i^3 u\, dx - \sum_{i=1}^{2}\int \partial_i^3 u_3\, \partial_3 u \cdot \partial_i^3 u\, dx,$$
$$= I_{1,2,1} + I_{1,2,2}.$$

An argument similar to (3.5) and (3.4) yields

$$I_{1,2,1} \le C\sum_{i=1}^{2}\sum_{k=1}^{2} \|\partial_i^k u_3\|_{L^2}^{\frac{1}{2}} \|\partial_i^k \partial_1 u_3\|_{L^2}^{\frac{1}{2}} \|\partial_i^{3-k}\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_i^{3-k}\partial_2\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_i^3\partial_3 u\|_{L^2}$$
$$\le C\|u\|_{H^{3,0}}\|\partial_3\nabla_h u\|_{H^{2,0}}^2,$$

and

$$I_{1,2,2} = \sum_{i=1}^{2}\int \partial_i^3\partial_3 u_3\, u \cdot \partial_i^3 u\, dx + \sum_{i=1}^{2}\int \partial_i^3 u_3\, u \cdot \partial_i^3\partial_3 u\, dx$$
$$\le C\sum_{i=1}^{2}\|\partial_i^3\partial_3 u\|_{L^2}\|u\|_{L^\infty}\|\partial_i^3 u\|_{L^2}$$
$$\le C\|\nabla_h^3\partial_3 u\|_{L^2}\|\partial_3 u\|_{H^{2,0}}\|\nabla_h^3 u\|_{L^2} \le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2. \tag{3.6}$$

where we have used integration by parts for $I_{1,2,2}$. Collecting all the estimates above yields

$$I_1 \le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2. \tag{3.7}$$

$I_2, I_3$, and $I_4$ can be dealt with similarly. For $I_2$, we first rewrite it as

$$I_2 = \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int \partial_i^k b \cdot \partial_i^{3-k}\nabla b \cdot \partial_i^3 u\, dx$$
$$= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int \partial_i^k b_h \cdot \partial_i^{3-k}\nabla_h b \cdot \partial_i^3 u\, dx + \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int \partial_i^k b_3\, \partial_i^{3-k}\partial_3 b \cdot \partial_i^3 u\, dx$$
$$:= I_{2,1} + I_{2,2}.$$

Furthermore, we split $I_{2,1}$ and $I_{2,2}$ in terms of the index $k$ to get

$$\begin{aligned} I_{2,1} &= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int \partial_i^k b_h \cdot \partial_i^{3-k}\nabla_h b \cdot \partial_i^3 u \, dx \\ &= I_{2,1,1} + I_{2,1,2} + I_{2,1,3}, \\ I_{2,2} &= \sum_{i=1}^{2}\sum_{k=1}^{2} C_3^k \int \partial_i^k b_3 \, \partial_i^{3-k}\partial_3 b \cdot \partial_i^3 u \, dx + \sum_{i=1}^{2}\int \partial_i^3 b_3 \, \partial_3 b \cdot \partial_i^3 u \, dx, \\ &= I_{2,2,1} + I_{2,2,2}. \end{aligned}$$

A direct application of (2.2) gives

$$\begin{aligned} I_{2,1,1} + I_{2,1,3} &\le C \sum_{i=1}^{2} \|\nabla_h b\|_{L^\infty} \|\nabla_h^3 b\|_{L^2} \|\partial_i^3 u\|_{L^2} \\ &\le C\|\nabla_h^3 b\|_{L^2} \big(\|\partial_3 \nabla_h^3 u\|_{L^2}^2 + \|\nabla_h \partial_3 b\|_{H^{2,0}}^2\big). \end{aligned}$$

As in (3.6), integration by parts and (2.2) yields

$$\begin{aligned} I_{2,2,2} &= -\sum_{i=1}^{2}\int \partial_i^3 \partial_3 b_3 \, b \cdot \partial_i^3 u \, dx - \sum_{i=1}^{2}\int \partial_i^3 b_3 \, b \cdot \partial_i^3 \partial_3 u \, dx \\ &\le \sum_{i=1}^{2} \big(\|\partial_i^3 \partial_3 b\|_{L^2} \|b\|_{L^\infty} \|\partial_i^3 u\|_{L^2} + \|\partial_i^3 b\|_{L^2} \|b\|_{L^\infty} \|\partial_i^3 \partial_3 u\|_{L^2}\big) \\ &\le C\|\nabla_h^3 \partial_3 b\|_{L^2} \|\partial_3 b\|_{H^{2,0}} \|u\|_{H^{3,0}} + C\|b\|_{H^{3,0}} \|\partial_3 b\|_{H^{2,0}} \|\partial_3 u\|_{H^{3,0}} \\ &\le C\|(u,b)\|_{H^{3,0}} \big(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2\big). \end{aligned}$$

By (2.4) and (2.1), $I_{2,1,2}$ and $I_{2,2,1}$ can be estimated as

$$\begin{aligned} I_{2,1,2} + I_{2,2,1} &= 3\sum_{i=1}^{2}\int \partial_i^2 b_h \cdot \nabla_h \partial_i b \cdot \partial_i^3 u \, dx + \sum_{i=1}^{2}\sum_{k=1}^{2} C_3^k \int \partial_i^k b_3 \, \partial_i^{3-k} \partial_3 b \cdot \partial_i^3 u \, dx \\ &\le C \sum_{i=1}^{2} \|\partial_i^2 b\|_{L^2}^{\frac12} \|\partial_i^2 \partial_1 b\|_{L^2}^{\frac12} \|\partial_i \nabla_h b\|_{L^2}^{\frac12} \|\partial_2 \partial_i \nabla_h b\|_{L^2}^{\frac12} \|\partial_i^3 \partial_3 u\|_{L^2} \\ &\quad + C \sum_{i=1}^{2}\sum_{k=1}^{2} \|\partial_i^k b_3\|_{L^2}^{\frac12} \|\partial_i^k \partial_1 b_3\|_{L^2}^{\frac12} \|\partial_i^{3-k} \partial_3 b\|_{L^2}^{\frac12} \|\partial_i^{3-k} \partial_2 \partial_3 b\|_{L^2}^{\frac12} \|\partial_i^3 \partial_3 u\|_{L^2} \\ &\le C\|b\|_{H^{3,0}} (\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2). \end{aligned}$$

Consequently,

$$I_2 \le C\|(u,b)\|_{H^{3,0}}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2). \tag{3.8}$$

Using a similar argument as in the estimates of $I_1$ and $I_2$, we can show

$$I_3 + I_4 \le C\|(u,b)\|_{H^{3,0}}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2). \tag{3.9}$$

For the convenience of the readers, we give some details. First, we still split them as

$$\begin{aligned}
I_3 + I_4 = &-\sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int (\partial_i^k u_h \cdot \partial_i^{3-k}\nabla_h b - \partial_i^k b_h \cdot \partial_i^{3-k}\nabla_h u)\cdot \partial_i^3 b\, dx \\
&-\sum_{i=1}^{2}\sum_{k=1}^{2} C_3^k \int (\partial_i^k u_3\, \partial_i^{3-k}\partial_3 b \cdot \partial_i^3 b - \partial_i^k b_3\, \partial_i^{3-k}\partial_3 u \cdot \partial_i^3 b) dx \\
&-\sum_{i=1}^{2} \int_{\mathbb{R}^3} (\partial_i^3 u_3\, \partial_3 b \cdot \partial_i^3 b - \partial_i^3 b_3\, \partial_3 u \cdot \partial_i^3 b)\, dx \\
&= I_{34,1,1} + I_{34,1,2} + I_{34,1,3} + I_{34,2,1} + I_{34,2,2}.
\end{aligned}$$

where $I_{34,1,k}$ represents three terms of the first integral term in terms of the derivative $k$. As in $I_{1,1}$, $I_{34,1,1}$ through $I_{34,1,3}$ can be bounded by

$$\begin{aligned}
I_{34,1,1} + I_{34,1,3} \le\ & C\sum_{i=1}^{2}\left(\|\partial_i u\|_{L^\infty}\|\partial_i^2\nabla_h b\|_{L^2} + \|\partial_i b\|_{L^\infty}\|\partial_i^2\nabla_h u\|_{L^2}\right)\|\partial_i^3 b\|_{L^2} \\
&+ C\sum_{i=1}^{2}\left(\|\partial_i^3 u\|_{L^2}\|\nabla_h b\|_{L^\infty} + C\|\partial_i^3 b\|_{L^2}\|\nabla_h u\|_{L^\infty}\right)\|\partial_i^3 b\|_{L^2} \\
\le\ & C\|(u,b)\|_{H^{3,0}}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2), \\
I_{34,1,2} \le\ & C\sum_{i=1}^{2}\|\partial_i^2 u\|_{L^2}^{\frac12}\|\partial_i^2\partial_1 u\|_{L^2}^{\frac12}\|\partial_i\nabla_h b\|_{L^2}^{\frac12}\|\partial_i\nabla_h\partial_2 b\|_{L^2}^{\frac12}\|\partial_i^3\partial_3 b\|_{L^2} \\
&+ C\sum_{i=1}^{2}\|\partial_i^2 b\|_{L^2}^{\frac12}\|\partial_i^2\partial_1 b\|_{L^2}^{\frac12}\|\partial_i\nabla_h u\|_{L^2}^{\frac12}\|\partial_i\partial_2\nabla_h u\|_{L^2}^{\frac12}\|\partial_i^3\partial_3 b\|_{L^2} \\
\le\ & C\|(u,b)\|_{H^{3,0}}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2).
\end{aligned}$$

Similarly to $I_{1,2}$, we have

$$I_{34,2,1} \le C\sum_{i=1}^{2}\sum_{k=1}^{2}\|\partial_i^k u_3\|_{L^2}^{\frac12}\|\partial_i^k\partial_1 u_3\|_{L^2}^{\frac12}\|\partial_i^{3-k}\partial_3 b\|_{L^2}^{\frac12}\|\partial_i^{3-k}\partial_2\partial_3 b\|_{L^2}^{\frac12}\|\partial_i^3\partial_3 b\|_{L^2}$$
$$+ C\sum_{i=1}^{2}\sum_{k=1}^{2}\|\partial_i^k b_3\|_{L^2}^{\frac12}\|\partial_i^k\partial_1 b_3\|_{L^2}^{\frac12}\|\partial_i^{3-k}\partial_3 u\|_{L^2}^{\frac12}\|\partial_i^{3-k}\partial_2\partial_3 u\|_{L^2}^{\frac12}\|\partial_i^3\partial_3 b\|_{L^2}$$
$$\le C\|(u,b)\|_{H^{3,0}}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2)$$

and

$$I_{34,2,2} = \sum_{i=1}^{2}\int (\partial_i^3\partial_3 u_3\ \ b\cdot\partial_i^3 b + \partial_i^3 u_3\ \ b\cdot\partial_i^3\partial_3 b)\,dx$$
$$-\sum_{i=1}^{2}\int (\partial_i^3\partial_3 b_3\ u\cdot\partial_i^3 b + \partial_i^3 b_3\ u\cdot\partial_i^3\partial_3 b)\,dx$$
$$\le C\sum_{i=1}^{2}\big(\|\partial_i^3\partial_3 u\|_{L^2}\|\partial_i^3 b\|_{L^2} + \|\partial_i^3 u\|_{L^2}\|\partial_i^3\partial_3 b\|_{L^2}\big)\|b\|_{L^\infty}$$
$$+ C\sum_{i=1}^{2}\big(\|\partial_i^3\partial_3 b\|_{L^2}\|\partial_i^3 b\|_{L^2} + \|\partial_i^3 b\|_{L^2}\|\partial_i^3\partial_3 b\|_{L^2}\big)\|u\|_{L^\infty}$$
$$\le C\|(u,b)\|_{H^{3,0}}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2),$$

which, together with the estimate for $I_{34,1,1}$ through $I_{34,1,3}$, gives the desired bound (3.9). Inserting (3.7), (3.8), and (3.9) in (3.3) and combining with (3.2), we obtain

$$\frac12\frac{d}{dt}\Big(\|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2}\|(\partial_i^3 u,\partial_i^3 b)\|_{L^2}^2\Big) + c_0(\nu\|\partial_3 u\|_{H^{3,0}}^2 + \eta\|\partial_3 b\|_{H^{3,0}}^2)$$
$$\le C\|(u,b)\|_{H^{3,0}}\big(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2\big), \tag{3.10}$$

where we have used the fact that $\|\partial_3 v\|_{H^{3,0}}$ is equivalent to

$$\|\partial_3 v\|_{L^2} + \sum_{i=1}^{2}\|\partial_i^3\partial_3 v\|_{L^2}$$

in $\Omega$. Then integrating (3.10) over $[0,t]$ for any $t > 0$ yields the desired estimate. This completes the proof of Proposition 3.1. ■

### 3.2 Estimates for $\|(u_t, b_t)\|_{H^1}$

The first subsection has obtained an energy inequality involving $(u,b)$ in $H^{3,0}$, i.e., the horizontal derivatives of $(u,b)$. To establish the well-posedness in $H^3$, we also need a bound for $\|(\partial_3 u, \partial_3 b)\|_{H^2}$. Unfortunately, $\|(\partial_3 u, \partial_3 b)\|_{H^2}$ can not be estimated directly due to the lack of boundary condition on the vertical derivatives of $(u,b)$. A key observation is to resort to the elliptic regularity theory and the Stokes estimates to achieve the goal. To do so, we need to establish an upper bound on $\|(u_t, b_t)\|_{H^1}$. This is shown by Propositions 3.2 and 3.3 below. To shorten the notation, we sometimes write $f_t$ for $\partial_t f$.

**Proposition 3.2.** Let $(u,b)$ be the solution of the system (1.1). Then, for some constant $C > 0$,

$$\|u_t\|_{L^2} + \|b_t\|_{L^2} \le Ce^{C\int_0^t \|(\partial_3 u, \partial_3 b)(\tau)\|_{H^{3,0}}^2 d\tau}\left(\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}\right). \tag{3.11}$$

**Proof of Proposition 3.2** Applying $\partial_t$ to the system (1.1) yields

$$\begin{cases} \partial_{tt} u + ((u\cdot\nabla)u)_t + \nabla p_t = \nu\partial_{33} u_t + ((b\cdot\nabla)b)_t, \\ \partial_{tt} b + ((u\cdot\nabla)b)_t = \eta\partial_{33} b_t + ((b\cdot\nabla)u)_t, \\ \nabla\cdot u_t = \nabla\cdot b_t = 0, \\ u_t|_{\partial\Omega} = b_t|_{\partial\Omega} = 0. \end{cases} \tag{3.12}$$

Taking the $L^2$-inner product of (3.12) with $(u_t, b_t)$ and applying the boundary conditions $u_t|_{\partial\Omega} = b_t|_{\partial\Omega} = 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u_t,\ b_t)\|_{L^2}^2 + \left(\nu\|\partial_3 u_t\|_{L^2}^2 + \eta\|\partial_3 b_t\|_{L^2}^2\right) \\ &= -\int (u\cdot\nabla u)_t\cdot u_t dx + \int (b\cdot\nabla b)_t\cdot u_t dx - \int (u\cdot\nabla b)_t\cdot b_t dx + \int (b\cdot\nabla u)_t\cdot b_t dx \\ &:= J_1 + \cdots + J_4. \end{aligned} \tag{3.13}$$

By $u_t|_{\partial\Omega} = 0$, integration by parts, and the inequalities (2.2) and (2.6),

$$\begin{aligned} J_1 &= -\int \partial_t u_h \cdot \nabla_h u \cdot u_t dx - \int \partial_t u_3\, \partial_3 u \cdot u_t dx \\ &\le \|\nabla_h u\|_{L^\infty} \|u_t\|_{L^2}^2 + C\|\partial_3 u_t\|_{L^2} \|\partial_3 u\|_{L^2}^{\frac14} \|\partial_1 \partial_3 u\|_{L^2}^{\frac14} \|\partial_2 \partial_3 u\|_{L^2}^{\frac14} \|\partial_1 \partial_2 \partial_3 u\|_{L^2}^{\frac14} \|u_t\|_{L^2} \\ &\le C\|\partial_3 \nabla_h u\|_{H^{2,0}} \|u_t\|_{L^2} \|\partial_3 u_t\|_{L^2} + C\|\partial_3 u_t\|_{L^2} \|\partial_3 u\|_{H^{2,0}} \|u_t\|_{L^2} \\ &\le C\|\partial_3 u\|_{H^{3,0}}^2 \|u_t\|_{L^2}^2 + \frac{\nu}{4} \|\partial_3 u_t\|_{L^2}^2, \end{aligned} \tag{3.14}$$

where we have used the fact $\|u_t\|_{L^2} \le C\|\partial_3 u_t\|_{L^2}$ (the Poincaré-type inequality (2.1)). With a minor modification of (3.14), we get

$$J_3 \le C\|\partial_3 b\|_{H^{3,0}}^2 \|b_t\|_{L^2}^2 + \frac{\nu}{4} \|\partial_3 u_t\|_{L^2}^2. \tag{3.15}$$

Also,

$$\begin{aligned} J_2 + J_4 &= \int b_t \cdot \nabla b \cdot u_t dx + \int b_t \cdot \nabla u \cdot b_t dx \\ &\le C(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2) \|(u_t, b_t)\|_{L^2}^2 + \frac{\eta}{2} \|\partial_3 b_t\|_{L^2}^2. \end{aligned} \tag{3.16}$$

Substituting (3.14), (3.15), and (3.16) into (3.13), we obtain

$$\frac{d}{dt} \|(u_t, b_t)\|_{L^2}^2 + \left(\nu \|\partial_3 u_t\|_{L^2}^2 + \eta \|\partial_3 b_t\|_{L^2}^2\right) \le C(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2) \|(u_t, b_t)\|_{L^2}^2.$$

Then, Gronwall's inequality implies

$$\|u_t\|_{L^2} + \|b_t\|_{L^2} \le C e^{C \int_0^t \|(\partial_3 u, \partial_3 b)(\tau)\|_{H^{3,0}}^2 d\tau} \left(\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}\right).$$

Here we have used

$$\|(u_t(0), b_t(0)\|_{L^2} \le C\left(\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}\right),$$

which follows from

$$\begin{aligned} \partial_t u(0) &= -u_0 \cdot \nabla u_0 - \nabla p_0 + \partial_3^2 u_0 + b_0 \cdot \nabla b_0, \\ \partial_t b(0) &= -u_0 \cdot \nabla b_0 + \partial_3^2 b_0 + b_0 \cdot \nabla u_0, \end{aligned}$$

with the pressure determined by the elliptic equations

$$-\Delta p_0 = \nabla\cdot(u_0\cdot\nabla u_0 - b_0\cdot\nabla b_0),\ x\in\Omega,\quad \nabla p_0\cdot n = \partial_3^2 u_0\cdot n,\ x\in\partial\Omega.$$

This completes the proof of Proposition 3.2. ■

**Proposition 3.3.** Let $(u, b)$ be the solution of the system (1.1). Then, for some constant $C > 0$,

$$\|\nabla u_t\|_{L^2} + \|\nabla b_t\|_{L^2} \le Ce^{C\int_0^t \left(1+\|(\nabla u,\nabla b)(\tau)\|_{L^2}^2\right)\|(\partial_3 u,\partial_3 b)(\tau)\|_{H^{3,0}}^2 d\tau} \times \left(\|(u_0, b_0)\|_{H^2}^2 + \|(\nabla\partial_3^2 u_0, \nabla\partial_3^2 b_0)\|_{L^2}\right). \tag{3.17}$$

**Proof of Proposition 3.3** Taking the $L^2$-inner product of (3.12) with $(u_{tt}, b_{tt})$ and $(\Delta_h u_t, \Delta_h b_t)$, respectively, integrating by parts, and applying the boundary conditions for $u, b$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\nu\|\partial_3 u_t\|_{L^2}^2 + \eta\|\partial_3 b_t\|_{L^2}^2 + \|(\nabla_h u_t, \nabla_h b_t)\|_{L^2}^2)\\
&\quad + \left(\|(u_{tt}, b_{tt})\|_{L^2}^2 + \nu\|\partial_3\nabla_h u_t\|_{L^2}^2 + \eta\|\partial_3\nabla_h b_t\|_{L^2}^2\right)\\
&= -\int (u\cdot\nabla u)_t\cdot u_{tt}\,dx + \int (b\cdot\nabla b)_t\cdot u_{tt}\,dx - \int (u\cdot\nabla b)_t\cdot b_{tt}\,dx\\
&\quad + \int (b\cdot\nabla u)_t\cdot b_{tt}\,dx + \int (u\cdot\nabla u)_t\cdot\Delta_h u_t\,dx - \int (b\cdot\nabla b)_t\cdot\Delta_h u_t\,dx\\
&\quad + \int (u\cdot\nabla b)_t\cdot\Delta_h b_t\,dx - \int (b\cdot\nabla u)_t\cdot\Delta_h b_t\,dx\\
&:= K_1 + \cdots + K_8,
\end{aligned} \tag{3.18}$$

where we have used

$$\int \nabla p_t\cdot u_{tt}\,dx = 0 \quad\text{and}\quad \int \nabla p_t\cdot\Delta_h u_t\,dx = 0.$$

Firstly, $K_1$ can be written as

$$K_1 = -\int u_t\cdot\nabla u\cdot u_{tt}\,dx - \int u\cdot\nabla u_t\cdot u_{tt}\,dx.$$

By the anisotropic inequalities (2.5), (2.1), and (2.2),

$$\begin{aligned} K_1 &\le C\|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_2 u_t\|_{L^2}^{\frac14}\|\partial_2\partial_3 u_t\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2} + C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^2}\\ &\le C\|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_2\partial_3 u_t\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2} + C\|\partial_3 u\|_{H^{2,0}}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^2}\\ &\le C(1+\|\nabla u\|_{L^2}^2)\|\partial_3 u\|_{H^{2,0}}^2\|\nabla u_t\|_{L^2}^2 + \Big(\frac{\nu}{10}\|\partial_2\partial_3 u_t\|_{L^2}^2 + \frac14\|u_{tt}\|_{L^2}^2\Big). \end{aligned}$$

where we also have used

$$\|\partial_1\nabla u\|_{L^2}^{\frac12} \le C\|\partial_3\partial_1 u\|_{H^{1,0}}^{\frac12}.$$

Similarly, we have

$$\begin{aligned} K_2 &\le C(1+\|\nabla b\|_{L^2}^2)\|\partial_3 b\|_{H^{2,0}}^2\|\nabla b_t\|_{L^2}^2 + \Big(\frac{\eta}{8}\|\partial_2\partial_3 b_t\|_{L^2}^2 + \frac14\|u_{tt}\|_{L^2}^2\Big),\\ K_3 &\le C\|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_2\partial_3 u_t\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1\nabla b\|_{L^2}^{\frac12}\|b_{tt}\|_{L^2} + C\|\partial_3 u\|_{H^{2,0}}\|\nabla b_t\|_{L^2}\|b_{tt}\|_{L^2}\\ &\le C(1+\|\nabla b\|_{L^2}^2)\|(\partial_3 u,\partial_3 b)\|_{H^{2,0}}^2\|(\nabla u_t,\nabla b_t)\|_{L^2}^2 + \Big(\frac{\nu}{10}\|\partial_2\partial_3 u_t\|_{L^2}^2 + \frac14\|b_{tt}\|_{L^2}^2\Big),\\ K_4 &\le C\|\partial_3 b_t\|_{L^2}^{\frac12}\|\partial_2\partial_3 b_t\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|b_{tt}\|_{L^2} + C\|\partial_3 b\|_{H^{2,0}}\|\nabla u_t\|_{L^2}\|b_{tt}\|_{L^2}\\ &\le C(1+\|\nabla u\|_{L^2}^2)\|(\partial_3 u,\partial_3 b)\|_{H^{2,0}}^2\|(\nabla u_t,\nabla b_t)\|_{L^2}^2 + \Big(\frac{\eta}{8}\|\partial_2\partial_3 b_t\|_{L^2}^2 + \frac14\|b_{tt}\|_{L^2}^2\Big). \end{aligned}$$

To deal with $K_5$, we decompose it as

$$\begin{aligned} K_5 &= -\int \nabla_h u_t\cdot\nabla u\cdot\nabla_h u_t\,dx - \int u_t\cdot\nabla\nabla_h u\cdot\nabla_h u_t\,dx - \int \nabla_h u\cdot\nabla u_t\cdot\nabla_h u_t\,dx\\ &:= K_{5,1}+K_{5,2}+K_{5,3}. \end{aligned} \tag{3.19}$$

Applying (2.6), (2.4), and (2.2) to $K_{5,1}$, $K_{5,2}$, and $K_{5,3}$, respectively, and combining with (2.1), we get

$$\begin{aligned} K_{51} &\le C\|\partial_3\nabla_h u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac14}\|\nabla_h u_t\|_{L^2}\\ &\le C\|\partial_3 u\|_{H^{3,0}}^2\|\nabla u_t\|_{L^2}^2 + \frac{\nu}{30}\|\partial_3\nabla_h u_t\|_{L^2}^2, \end{aligned} \tag{3.20}$$

$$\begin{aligned} K_{52} &\le C\|u_t\|_{L^2}^{\frac12}\|\partial_1 u_t\|_{L^2}^{\frac12}\|\nabla\nabla_h u\|_{L^2}^{\frac12}\|\partial_2\nabla\nabla_h u\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_t\|_{L^2}\\ &\le C\|\partial_3\nabla_h u\|_{H^{2,0}}^2\|\nabla u_t\|_{L^2}^2 + \frac{\nu}{30}\|\partial_3\nabla_h u_t\|_{L^2}^2, \end{aligned}$$

and

$$K_{53} \le \|\nabla_h u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla_h u_t\|_{L^2} \le C\|\partial_3\nabla_h u\|_{H^{2,0}}\|\nabla u_t\|_{L^2}\|\partial_3\nabla_h u_t\|_{L^2}$$
$$\le C\|\partial_3\nabla_h u\|_{H^{2,0}}^2\|\nabla u_t\|_{L^2}^2 + \frac{\nu}{30}\|\partial_3\nabla_h u_t\|_{L^2}^2. \tag{3.21}$$

Thus,

$$K_5 \le C\|\partial_3 u\|_{H^{3,0}}^2\|\nabla u_t\|_{L^2}^2 + \frac{\nu}{10}\|\partial_3\nabla_h u_t\|_{L^2}^2.$$

By a similar argument as the one for $K_5$, we can show the estimates for the rest of terms in (3.18). First,

$$K_6 + K_8 = \int \nabla_h b_t \cdot \nabla b \cdot \nabla_h u_t + \int b_t \cdot \nabla\nabla_h b \cdot \nabla_h u_t + \int \nabla_h b \cdot \nabla b_t \cdot \nabla_h u_t$$
$$+ \int \nabla_h b_t \cdot \nabla u \cdot \nabla_h b_t + \int b_t \cdot \nabla\nabla_h u \cdot \nabla_h b_t + \int \nabla_h b \cdot \nabla u_t \cdot \nabla_h b_t. \tag{3.22}$$

Then invoking (3.20), we obtain

$$\int \nabla_h b_t \cdot \nabla b \cdot \nabla_h u_t + \int \nabla_h b_t \cdot \nabla u \cdot \nabla_h b_t$$
$$\le C\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2\|(\nabla u_t, \nabla b_t)\|_{L^2}^2 + \frac{\eta}{24}\|\partial_3\nabla_h b_t\|_{L^2}^2.$$

Similarly to $K_{52}$,

$$\int b_t \cdot \nabla\nabla_h b \cdot \nabla_h u_t + \int b_t \cdot \nabla\nabla_h u \cdot \nabla_h b_t$$
$$\le C\|(\partial_3\nabla_h u, \partial_3\nabla_h b)\|_{H^{2,0}}^2\|\nabla b_t\|_{L^2}^2 + \left(\frac{\nu}{20}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac{\eta}{24}\|\partial_3\nabla_h b_t\|_{L^2}^2\right).$$

Also, by (3.21)

$$\int \nabla_h b \cdot \nabla b_t \cdot \nabla_h u_t + \int \nabla_h b \cdot \nabla u_t \cdot \nabla_h b_t$$
$$\le C\|\partial_3\nabla_h b\|_{H^{2,0}}^2\|(\nabla u_t, \nabla b_t)\|_{L^2}^2 + \left(\frac{\nu}{20}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac{\eta}{24}\|\partial_3\nabla_h b_t\|_{L^2}^2\right).$$

Consequently, we have

$$K_6 + K_8 \le C\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2\|(\nabla u_t, \nabla b_t)\|_{L^2}^2 + \left(\frac{\nu}{10}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac{\eta}{8}\|\partial_3\nabla_h b_t\|_{L^2}^2\right).$$

Analogously,

$$K_7 = -\int \nabla_h u_t \cdot \nabla b \cdot \nabla_h b_t \, dx - \int u_t \cdot \nabla\nabla_h b \cdot \nabla_h b_t \, dx - \int \nabla_h u \cdot \nabla b_t \cdot \nabla_h b_t \, dx$$
$$\le C\|(\partial_3 u, \partial_3 b)\|^2_{H^{3,0}} \|(\nabla u_t, \nabla b_t)\|^2_{L^2} + \Big(\frac{\nu}{10}\|\partial_3 \nabla_h u_t\|^2_{L^2} + \frac{\eta}{8}\|\partial_3 \nabla_h b_t\|^2_{L^2}\Big).$$

Substituting all the estimates above for $K_1$ through $K_8$ into (3.18), we have

$$\frac{d}{dt}\Big(\nu\|\partial_3 u_t\|^2_{L^2} + \eta\|\partial_3 b_t\|^2_{L^2} + \|(\nabla_h u_t, \nabla_h b_t)\|^2_{L^2}\Big)$$
$$+ (\|(u_{tt}, b_{tt})\|^2_{L^2} + \nu\|\partial_3 \nabla_h u_t\| + \eta\|\partial_3 \nabla_h b_t)\|^2_{L^2})$$
$$\le C(1 + \|(\nabla u, \nabla b)\|^2_{L^2})\|(\partial_3 u, \partial_3 b)\|^2_{H^{3,0}} \|(\nabla u_t, \nabla b_t)\|^2_{L^2}.$$

Using the inequality $\|u_0 \cdot \nabla u_0\|_{H^1} \le C\,\|u_0\|^2_{H^2}$ in the equation of $(u_t(0), b_t(0))$, we have

$$\|(u_t(0), b_t(0))\|_{H^1} \le C\big(\|(u_0, b_0)\|^2_{H^2} + \|(\nabla \partial_3^2 u_0, \nabla \partial_3^2 b_0)\|_{L^2}\big). \tag{3.23}$$

Gronwall's inequality with (3.23) then leads to the desired estimate of Proposition 3.3■

### 3.3 Estimates for $\|(\partial_3 u, \partial_3 b)\|_{H^2}$ and $\|(u, b)\|_{H^3}$

This subsection presents the estimates for $\|(\partial_3 u, \partial_3 b)\|_{H^2}$ and thus for $\|(u, b)\|_{H^3}$. The approach here is to invoke the elliptic regularity theory and the Stokes estimates. For the sake of clarity, we state the results in two propositions with Proposition 3.4 containing the $H^2$-bound and Proposition 3.5 the $H^3$-bound.

**Proposition 3.4.** Let $(u, b)$ be the solution to the system (1.1). Then, we have

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} + \|b\|_{H^2} \le C\big(\|(\partial_t u, \partial_t b)\|_{L^2} + \|(\nabla_h u, \nabla_h b)\|_{H^{1,0}} (\|\nabla u\|_{H^1} + \|\nabla b\|_{H^1})$$
$$+ \|(\Delta_h u, \Delta_h b)\|_{L^2}\big). \tag{3.24}$$

**Proof of Proposition 3.4** We can then rewrite the velocity equation and magnetic equation of (1.1) as

$$\begin{cases} -\Delta u + \nabla p = f, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{3.25}$$

and

$$\begin{cases} -\Delta b = g, & x \in \Omega,\ t > 0, \\ b(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ \nabla \cdot b = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{3.26}$$

respectively, with

$$f := -\partial_t u - (u\cdot\nabla)u + (b\cdot\nabla)b - \Delta_h u,$$
$$g := -\partial_t b - (u\cdot\nabla)b + (b\cdot\nabla)u - \Delta_h b.$$

It follows from the Stokes estimates (2.8) that

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} \le C\big(\|\partial_t u\|_{L^2} + \|(u\cdot\nabla)u\|_{L^2} + \|(b\cdot\nabla)b\|_{L^2} + \|\Delta_h u\|_{L^2}\big).$$

By Hölder's inequality, (2.3) and Sobolev's imbedding inequality,

$$\begin{aligned} \|(u\cdot\nabla)u\|_{L^2} \le& \|u_h\|_{L^4}\|\nabla_h u\|_{L^4} + \|u_3\|_{L^4}\|\partial_3 u\|_{L^4} \\ \le& C\|\nabla_h u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}} \\ &+ C\|\nabla_h u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{H^1} \\ \le& C\|\nabla_h u\|_{H^{1,0}}\|\nabla u\|_{H^1}, \end{aligned} \tag{3.27}$$

where we have used the divergence-free condition for $u$. Similarly,

$$\|(b\cdot\nabla)b\|_{L^2} \le C\|\nabla_h b\|_{H^{1,0}}\|\nabla b\|_{H^1}.$$

Thus,

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} \le C(\|\partial_t u\|_{L^2} + \|\nabla_h u\|_{H^{1,0}}\|\nabla u\|_{H^1} + \|\nabla_h b\|_{H^{1,0}}\|\nabla b\|_{H^1} + \|\Delta_h u\|_{L^2}). \tag{3.28}$$

The bound for $b$ can be obtained by applying the classical elliptic regularity theory,

$$\|b\|_{H^2} \le C\Big(\|\partial_t b\|_{L^2} + \|(\nabla_h u, \nabla_h b)\|_{H^{1,0}}\|(\nabla u, \nabla b)\|_{H^1} + \|\Delta_h b\|_{L^2}\Big) \tag{3.29}$$

after we have applied the bounds

$$\|(u\cdot\nabla)b\|_{L^2} \le C(\|\nabla_h u\|_{L^2} + \|\nabla_h^2 b\|_{L^2})\,(\|\nabla u\|_{L^2} + \|\nabla b\|_{H^1}),$$
$$\|(b\cdot\nabla)u\|_{L^2} \le C(\|\nabla_h b\|_{L^2} + \|\nabla_h^2 u\|_{L^2})\,(\|\nabla b\|_{L^2} + \|\nabla u\|_{H^1}),$$

which follows from a similar argument as (3.27). The estimates in (3.28) and (3.29) lead to the desired bound. This completes the proof of Proposition3.4. ■

**Proposition 3.5.** Let $(u, b)$ be the solution to the system (1.1). Then we have

$$\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} \le C(\|u_t\|_{H^1} + \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\Delta_h u\|_{H^{1,0}}),$$
$$\|b\|_{H^3} \le C(\|b_t\|_{H^1} + \|u\|_{H^2}\|\nabla b\|_{H^1} + \|b\|_{H^2}\|\nabla u\|_{H^1} + \|\Delta_h b\|_{H^{1,0}}).$$

**Proof of Proposition 3.5** The Stokes estimates applied to (3.25) yield

$$\begin{aligned}\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} &\le C(\|u_t\|_{H^1} + \|u\cdot\nabla u\|_{H^1} + \|b\cdot\nabla b\|_{H^1} + \|\Delta_h u\|_{H^1})\\ &\le C(\|u_t\|_{H^1} + \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\Delta_h u\|_{H^{1,0}} + \|\partial_3\Delta_h u\|_{L^2}).\end{aligned}$$

By integration by parts, Hölder's inequality and Young's inequality, the last term on the right side above can be estimated as

$$\begin{aligned}\|\partial_3\Delta_h u\|_{L^2} &= \left(\int \partial_3^2\nabla_h u\cdot\nabla_h\Delta_h u\,dx\right)^{\frac12} \le \|\partial_3^2\nabla_h u\|_{L^2}^{\frac12}\|\nabla_h^3 u\|_{L^2}^{\frac12}\\ &\le \frac{1}{2C}\|\partial_3^2\nabla_h u\|_{L^2} + \frac{C}{2}\|\nabla_h^3 u\|_{L^2}.\end{aligned}\tag{3.30}$$

Therefore,

$$\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} \le C(\|u_t\|_{H^1} + \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\Delta_h u\|_{H^{1,0}}).$$

Next we show the estimate for $\|\nabla b\|_{H^2}$. Applying the classical elliptic regularity theory to the equation (3.26) and using Sobolev's inequality, we have

$$\begin{aligned}\|b\|_{H^3} &\le C(\|b_t\|_{H^1} + \|u\cdot\nabla b\|_{H^1} + \|b\cdot\nabla u\|_{H^1} + \|\Delta_h b\|_{H^1})\\ &\le C(\|b_t\|_{H^1} + \|u\|_{L^\infty}\|\nabla b\|_{H^1} + \|b\|_{L^\infty}\|\nabla u\|_{H^1} + \|\Delta_h b\|_{H^{1,0}} + \|\partial_3\Delta_h b\|_{L^2})\\ &\le C(\|b_t\|_{H^1} + \|u\|_{H^2}\|\nabla b\|_{H^1} + \|b\|_{H^2}\|\nabla u\|_{H^1} + \|\Delta_h b\|_{H^{1,0}} + \|\partial_3\Delta_h b\|_{L^2}).\end{aligned}$$

As in (3.30), we also have

$$\|\partial_3\Delta_h b\|_{L^2} \leq \frac{1}{2C}\|\partial_3^2\nabla_h b\|_{L^2} + \frac{C}{2}\|\nabla_h^3 b\|_{L^2}.$$

Then, we obtain

$$\|b\|_{H^3} \leq C(\|b_t\|_{H^1} + \|u\|_{H^2}\|\nabla b\|_{H^1} + \|b\|_{H^2}\|\nabla u\|_{H^1} + \|\Delta_h b\|_{H^{1,0}}).$$

This completes the proof of Proposition 3.5. ■

### 3.4 Proof of the global well-posedness part of Theorem 1.1

This subsection completes the proof of the global well-posedness part of Theorem 1.1 by combining the energy estimates obtained in the previous three subsections.

**Proof of the global well-posedness.** First we apply the bootstrapping argument to (3.1) in Proposition 3.1 to establish a global bound for $\|(u,b)\|_{H^{3,0}}$ and the time integral of $\|\partial_3 u(t)\|_{H^{3,0}}^2 + \|\partial_3 b(t)\|_{H^{3,0}}^2$ under the condition that the initial $H^{3,0}$-norm is sufficiently small.

Denoting

$$E(t) = \sup_{0\leq\tau\leq t}(\|u(\tau)\|_{H^{3,0}}^2 + \|b(\tau)\|_{H^{3,0}}^2) + \int_0^t (\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2)\,d\tau,$$

we obtain from (3.1) that

$$E(t) \leq a_0 E(0) + a_1 E^{\frac{3}{2}}(t) \tag{3.31}$$

for some constants $a_0 > 0$ and $a_1 > 0$. We assume that the initial norm is sufficiently small, say

$$\|(u_0,b_0)\|_{H^{3,0}} \leq \delta \leq \frac{1}{2}\sqrt{\frac{M}{a_0}}. \tag{3.32}$$

To apply the bootstrapping argument, we make the ansatz that

$$E(t) \leq M := \frac{1}{4a_1^2}. \tag{3.33}$$

Our goal is to show that $E(t)$ actually admits a smaller bound, say

$$E(t) \le \frac{M}{2}.$$

By (3.31), (3.32), and (3.33),

$$E(t) \le a_0 E(0) + a_1 E^{\frac{1}{2}}(t) E(t) \le a_0 \delta^2 + \frac{1}{2} E(t),$$

or

$$E(t) \le 2a_0 \delta^2 \le \frac{M}{2}, \tag{3.34}$$

which shows that $E(t)$ actually admits a smaller bound. The bootstrapping argument then asserts that (3.34) holds for any time, namely,

$$(\|u\|_{H^{3,0}}^2 + \|b\|_{H^{3,0}}^2) + \int_0^t (\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\partial_3 b(\tau)\|_{H^{3,0}}^2)\, d\tau \le C\delta^2. \tag{3.35}$$

Next we combine (3.35) with the energy estimates in Propositions 3.2 through 3.5 to establish a global bound for $\|(\partial_3 u, \partial_3 u)\|_{H^2}$ and thus $\|(u, b)\|_{H^3}$. It follows from (3.11) that

$$\begin{aligned} \|u_t\|_{L^2} + \|b_t\|_{L^2} &\le C e^{C \int_0^t \|(\partial_3 u, \partial_3 b)(\tau)\|_{H^{3,0}}^2 d\tau} (\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}) \\ &\le C\, (\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}), \end{aligned} \tag{3.36}$$

for a uniform constant $C$ (independent of $\delta$). Invoking the estimate (3.24) along with (3.35), we have

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} + \|b\|_{H^2} \le C\Big(\|(u_t, b_t)\|_{L^2} + \delta(\|\nabla u\|_{H^1} + \|\nabla b\|_{H^1}) + \|(\Delta_h u, \Delta_h b)\|_{L^2}\Big).$$

Then, for $\delta$ sufficiently small, we find

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} + \|b\|_{H^2} \le C(\|(u_t, b_t)\|_{L^2} + \|(\Delta_h u, \Delta_h b)\|_{L^2}), \tag{3.37}$$

which, together with (3.36) and (3.35), implies

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} + \|b\|_{H^2} \le C(\delta + \delta^2 + \|(\partial_3 u_0, \partial_3 b_0)\|_{H^1}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}) \tag{3.38}$$

for some constant $C > 0$. Furthermore, by (3.17), we obtain the uniform bound for $\|(\nabla u_t, \nabla b_t)\|_{L^2}$,

$$\|\nabla u_t\|_{L^2} + \|\nabla b_t\|_{L^2} \le C(\|(u_0, b_0)\|_{H^2}^2 + \|(\nabla\partial_3^2 u_0, \nabla\partial_3^2 b_0)\|_{L^2}). \tag{3.39}$$

As a consequence, by Proposition3.5, (3.35), (3.36), (3.38), and (3.39), we derive

$$\begin{aligned}\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} &\le C(\|u_t\|_{H^1} + \|(u,b)\|_{H^2} + \|\Delta_h u\|_{H^{1,0}})\\ &\le C(\delta + \delta^2 + \|(\partial_3 u_0, \partial_3 b_0)\|_{H^1}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{H^1}),\\ \|b\|_{H^3} &\le C(\|b_t\|_{H^1} + \|u\|_{H^2} + \|\Delta_h b\|_{H^{1,0}})\\ &\le C(\delta + \delta^2 + \|(\partial_3 u_0, \partial_3 b_0)\|_{H^1}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{H^1}).\end{aligned}$$

This completes the proof of the global well-posedness part in Theorem 1.1. ■

## 4 The decay estimates

This section is devoted to proving the decay estimates in Theorem 1.1. This is accomplished in three steps. The first step establishes the exponential decay rate for $\|(u,b)\|_{H^{3,0}}$ and $\|(\partial_3 u, \partial_3 b)\|_{L^2}$. An energy inequality involving these norms is derived in Proposition 4.1 to serve this purpose. The second step shows the exponential decay rate for $\|(u_t, b_t)\|_{H^{1,0}}$ and $\|(\partial_3 u_t, \partial_3 b_t)\|_{L^2}$. This step involves a key energy inequality stated in Proposition 4.2. The final step applies the Stokes estimates and the elliptic regularity theory to obtain the decay rates for $\|(u,b)\|_{H^3}$ and $\|\nabla p\|_{H^1}$.

We start with the Proposition 4.1 and its proof.

**Proposition 4.1.** Assume $(u, b)$ is the solution of the system (1.1). Then,

$$\frac{d}{dt}\big(\nu\|\partial_3 u\|_{L^2}^2 + \eta\|\partial_3 b\|_{L^2}^2\big) + \big(\|u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2\big) \le C\|(u,b)\|_{H^{1,0}}^2\big(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2\big). \tag{4.1}$$

for some constant $C > 0$.

**Proof of Proposition 4.1** Taking the $L^2$-inner product of (1.1) with $(u_t, b_t)$, and using the boundary condition $u_t = b_t = 0$ on $\partial\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\big(\nu\|\partial_3 u\|_{L^2}^2 + \eta\|\partial_3 b\|_{L^2}^2\big) + (\|u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2)$$
$$= -\int u\cdot\nabla u\cdot u_t\,dx + \int b\cdot\nabla b\cdot u_t\,dx - \int u\cdot\nabla b\cdot b_t\,dx + \int b\cdot\nabla u\cdot b_t\,dx.$$

Invoking the anisotropic inequality (2.5) and (2.6) yields

$$\begin{aligned}
-\int u\cdot\nabla u\cdot u_t\,dx &= -\int u_h\cdot\nabla_h u\cdot u_t dx - \int u_3\,\partial_3 u\cdot u_t dx\\
&\le C\|u_h\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla_h u\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3\nabla_h u\|_{L^2}^{\frac{1}{4}}\|u_t\|_{L^2}\\
&\quad + C\|\partial_3 u_3\|_{L^2}\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2\partial_3 u\|_{L^2}^{\frac{1}{4}}\|u_t\|_{L^2}\\
&\le C\|u\|_{H^{1,0}}^2\|\partial_3 u\|_{H^{2,0}}^2 + \frac{1}{4}\|u_t\|_{L^2}^2,
\end{aligned}$$

where we have used the fact $\|\partial_2\nabla_h u\|_{L^2} \le C\|\partial_3\partial_2\nabla_h u\|_{L^2}$. With a similar argument, the other integrals can be bounded as

$$\begin{aligned}
\int b\cdot\nabla b\cdot u_t &\le C\|b\|_{H^{1,0}}^2\|\partial_3 b\|_{H^{2,0}}^2 + \frac{1}{4}\|u_t\|_{L^2}^2,\\
-\int u\cdot\nabla b\cdot b_t &\le C\|u\|_{H^{1,0}}^2\|\partial_3 b\|_{H^{2,0}}^2 + \frac{1}{4}\|b_t\|_{L^2}^2,\\
\int b\cdot\nabla u\cdot b_t &\le C\|b\|_{H^{1,0}}^2\|\partial_3 u\|_{H^{2,0}}^2 + \frac{1}{4}\|b_t\|_{L^2}^2.
\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\big(\nu\|\partial_3 u\|_{L^2}^2 + \eta\|\partial_3 b\|_{L^2}^2\big) + \big(\|u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2\big) \le C\|(u,b)\|_{H^{1,0}}^2\big(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2\big).$$

This completes the proof of Proposition 4.1. ∎

The second proposition presents a sharper estimate on $\|(u_t, b_t)\|_{H^1}$. A different estimate on $\|(u_t, b_t)\|_{H^1}$ was obtained in Proposition 3.2 and Proposition 3.3. This upper bound is needed to extract the desired decay rate.

**Proposition 4.2.** Let $(u,b)$ be the solution of the system (1.1). Then,

$$\frac{d}{dt}\Big(\|(u_t,b_t)\|_{H^{1,0}}^2+\nu\|\partial_3 u_t\|_{L^2}^2+\eta\|\partial_3 b_t\|_{L^2}^2\Big)+\Big(\nu\|\partial_3 u_t\|_{H^{1,0}}^2+\eta\|\partial_3 b_t\|_{H^{1,0}}^2+\|(u_{tt},b_{tt})\|_{L^2}^2\Big)$$
$$\leq C\Big(\|(\partial_3 u,\partial_3 b)\|_{L^2}+\|(u,b)\|_{H^{2,0}}\Big)\|(\partial_3\nabla_h u,\partial_3\nabla_h b)\|_{H^{2,0}}^2\|(\nabla u_t,\nabla b_t)\|_{L^2}^2 \tag{4.2}$$

for some constant $C>0$.

**Proof of Proposition 4.2** By (3.13) and (3.18), we have

$$\frac{1}{2}\frac{d}{dt}(\|(u_t,b_t)\|_{H^{1,0}}^2+\nu\|\partial_3 u_t\|_{L^2}^2+\eta\|\partial_3 b_t\|_{L^2}^2)+(\nu\|\partial_3 u_t\|_{H^{1,0}}^2+\eta\|\partial_3 b_t\|_{H^{1,0}}^2+\|(u_{tt},b_{tt})\|_{L^2}^2)$$
$$:=J_1+\cdots+J_4+K_1\cdots+K_8,$$

where $J_1$ through $J_4$ and $K_1$ through $K_8$ are defined as in (3.13) and (3.18), respectively. By the anisotropic inequality (2.4), $J_1$ can be bounded as

$$\begin{aligned}J_1&=-\int\partial_t u_h\cdot\nabla_h u\cdot u_t\,dx-\int\partial_t u_3\,\partial_3 u\cdot u_t\,dx\\&\leq C\|\partial_3 u_t\|_{L^2}\Big(\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_2\nabla_h u\|_{L^2}^{\frac12}+\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12}\Big)\|u_t\|_{L^2}^{\frac12}\|\partial_1 u_t\|_{L^2}^{\frac12}\\&\leq C\|\partial_3 u_t\|_{L^2}^{\frac32}\Big(\|\nabla_h u\|_{H^{1,0}}+\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12}\Big)\|\partial_1 u_t\|_{L^2}^{\frac12}\\&\leq C(\|\nabla_h u\|_{H^{1,0}}^2+\|\partial_3 u\|_{L^2}^2)\|\partial_3\nabla_h u\|_{H^{1,0}}^2\|\nabla u_t\|_{L^2}^2+\frac{\nu}{12}\|\partial_3 u_t\|_{L^2}^2,\end{aligned}$$

where we have used

$$\|u_t\|_{L^2}\leq C\|\partial_3 u_t\|_{L^2},\ \|\nabla_h u\|_{H^{1,0}}\leq C\|\partial_3\nabla_h u\|_{H^{1,0}}$$

due to $u_t=0$ and $\nabla_h u=0$ on $\partial\Omega$. Similarly, for $J_2$ through $J_4$, we have

$$\begin{aligned}J_2+J_3+J_4\leq{}&C\Big(\|(\nabla_h u,\nabla_h b)\|_{H^{1,0}}^2+\|(\partial_3 u,\partial_3 b)\|_{L^2}^2\Big)\|(\partial_3\nabla_h u,\partial_3\nabla_h b)\|_{H^{1,0}}^2\|(\nabla u_t,\nabla b_t)\|_{L^2}^2\\&+\Big(\frac{\nu}{12}\|\partial_3 u_t\|_{L^2}^2+\frac{\eta}{8}\|\partial_3 b_t\|_{L^2}^2\Big).\end{aligned}$$

Now we bound $K_1$ through $K_4$. By the anisotropic inequalities (2.5) and (2.6),

$$\begin{aligned}
K_1 &= -\int u_t\cdot\nabla u\cdot u_{tt}\,dx - \int u_h\cdot\nabla_h u_t\cdot u_{tt}\,dx - \int u_3\,\partial_3 u_t\cdot u_{tt}\,dx\\
&\le C\|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_2 u_t\|_{L^2}^{\frac14}\|\partial_2\partial_3 u_t\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2}\\
&+ C\|u_h\|_{L^2}^{\frac14}\|\partial_1 u_h\|_{L^2}^{\frac14}\|\partial_2 u_h\|_{L^2}^{\frac14}\|\partial_1\partial_2 u_h\|_{L^2}^{\frac14}\|\nabla_h u_t\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_t\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2}\\
&+ C\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_2 u_3\|_{L^2}^{\frac14}\|\partial_2\partial_3 u_3\|_{L^2}^{\frac14}\|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_1\partial_3 u_t\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2}.
\end{aligned}$$

Due to $\partial_2 u_t = 0, \partial_2 u = 0, \partial_1\partial_2 u_h = 0$ on $\partial\Omega$, we can apply the Poincaré-type inequality (2.1) to obtain

$$\begin{aligned}
K_1 &\le C\|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_2\partial_3 u_t\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2}\\
&+ C\|u\|_{H^1}^{\frac12}\|\partial_3\partial_2 u\|_{H^{1,0}}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_t\|_{L^2}^{\frac12}\|u_{tt}\|_{L^2}\\
&\le C(\|\nabla u\|_{L^2}^2\|\nabla_h\nabla u\|_{L^2}^2 + \|u\|_{H^1}^2\|\partial_3\nabla_h u\|_{H^{1,0}}^2)\|\nabla u_t\|_{L^2}^2 + \Big(\frac{\nu}{36}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac14\|u_{tt}\|_{L^2}^2\Big)\\
&\le C\|u\|_{H^1}^2\|\partial_3\nabla_h u\|_{H^{1,0}}^2\|\nabla u_t\|_{L^2}^2 + \Big(\frac{\nu}{36}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac14\|u_{tt}\|_{L^2}^2\Big),
\end{aligned}$$

where we have used $\|\nabla_h\nabla u\|_{L^2} \le C\|\partial_3\nabla_h u\|_{H^{1,0}}$ in the last inequality. Similarly,

$$\begin{aligned}
K_2 &\le C\|b\|_{H^1}^2\|\partial_3\nabla_h b\|_{H^{1,0}}^2\|\nabla b_t\|_{L^2}^2 + \Big(\frac{\eta}{24}\|\partial_3\nabla_h b_t\|_{L^2}^2 + \frac14\|u_{tt}\|_{L^2}^2\Big),\\
K_3 &\le C(\|\nabla b\|_{L^2}^2\|\nabla_h\nabla b\|_{L^2}^2\|\nabla u_t\|_{L^2}^2 + \|u\|_{H^1}^2\|\partial_3\nabla_h u\|_{H^{1,0}}^2\|\nabla b_t\|_{L^2}^2)\\
&\quad + \Big(\frac{\nu}{36}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac{\eta}{24}\|\partial_3\nabla_h b_t\|_{L^2}^2 + \frac14\|b_{tt}\|_{L^2}^2\Big),\\
K_4 &\le C(\|\nabla u\|_{L^2}^2\|\nabla_h\nabla u\|_{L^2}^2\|\nabla b_t\|_{L^2}^2 + \|b\|_{H^1}^2\|\partial_3\nabla_h b\|_{H^{1,0}}^2\|\nabla u_t\|_{L^2}^2)\\
&\quad + \Big(\frac{\nu}{36}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac{\eta}{24}\|\partial_3\nabla_h b_t\|_{L^2}^2 + \frac14\|b_{tt}\|_{L^2}^2\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
K_1+K_2+K_3+K_4 &\le C(\|u\|_{H^1}^2 + \|b\|_{H^1}^2)\|(\partial_3\nabla_h u, \partial_3\nabla_h b)\|_{H^{1,0}}^2\|(\nabla u_t, \nabla b_t)\|_{L^2}^2\\
&\quad + \Big(\frac{\nu}{12}\|\partial_3\nabla_h u_t\|_{L^2}^2 + \frac{\eta}{8}\|\partial_3\nabla_h b_t\|_{L^2}^2 + \frac12\|(u_{tt}, b_{tt})\|_{L^2}^2\Big).
\end{aligned}$$

Recalling (3.19), we have

$$K_5 = -\int \nabla_h u_t \cdot \nabla u \cdot \nabla_h u_t \, dx - \int u_t \cdot \nabla\nabla_h u \cdot \nabla_h u_t \, dx - \int \nabla_h u \cdot \nabla u_t \cdot \nabla_h u_t \, dx$$
$$:= K_{51} + K_{52} + K_{53}.$$

By (2.6) and Poincaré-type inequality (2.1),

$$\begin{aligned} K_{51} &\le C\|\partial_3 \nabla_h u_t\|_{L^2} \|\nabla u\|_{L^2}^{\frac14} \|\partial_2 \nabla u\|_{L^2}^{\frac14} \|\partial_1 \nabla u\|_{L^2}^{\frac14} \|\partial_1\partial_2 \nabla u\|_{L^2}^{\frac14} \|\nabla_h u_t\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^{\frac14} \|\partial_2 \nabla u\|_{L^2}^{\frac14} \|\partial_1\partial_3 u\|_{H^{2,0}}^{\frac12} \|\nabla_h u_t\|_{L^2}^{\frac12} \|\partial_3 \nabla_h u_t\|_{L^2}^{\frac32} \\ &\le C\|\nabla u\|_{L^2} \|u\|_{H^2} \|\partial_3 \nabla_h u\|_{H^{2,0}}^2 \|\nabla u_t\|_{L^2}^2 + \frac{\nu}{36} \|\partial_3 \nabla_h u_t\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} K_{53} &\le C\|\nabla_h u\|_{L^2}^{\frac14} \|\partial_1 \nabla_h u\|_{L^2}^{\frac14} \|\partial_2 \nabla_h u\|_{L^2}^{\frac14} \|\partial_1\partial_2 \nabla_h u\|_{L^2}^{\frac14} \|\nabla u_t\|_{L^2} \|\partial_3 \nabla_h u_t\|_{L^2} \\ &\le C\|\nabla_h u\|_{L^2}^{\frac14} \|\partial_1 \nabla_h u\|_{L^2}^{\frac14} \|\partial_2 \nabla_h u\|_{L^2}^{\frac14} \|\partial_1\partial_2 \nabla_h u\|_{L^2}^{\frac14} \|\nabla u_t\|_{L^2}^{\frac12} (\|\partial_3 u_t\|_{L^2}^{\frac12} + \|\partial_3 \nabla_h u_t\|_{L^2}^{\frac12}) \|\partial_3 \nabla_h u_t\|_{L^2} \\ &\le C\|\nabla_h u\|_{H^{1,0}}^2 \|\partial_3 \nabla_h^2 u\|_{H^{1,0}}^2 \|\nabla u_t\|_{L^2}^2 + \frac{\nu}{36} \|\partial_3 u_t\|_{H^{1,0}}^2, \end{aligned}$$

where we have used $\|\nabla_h u_t\|_{L^2}^{\frac12} \le C\|\partial_3 \nabla_h u_t\|_{L^2}^{\frac12}$ in the second inequality. The estimates for $K_{52}$ is more complicated. By integration by parts, (2.4) and (2.5) and invoking $\|u_t\|_{L^2} \le C\|\partial_3 u_t\|_{L^2}$ and $\|\nabla_h u_t\|_{L^2} \le C\|\nabla_h \partial_3 u_t\|_{L^2}$, we deduce

$$\begin{aligned} K_{52} &= -\int \partial_t u_h \cdot \nabla_h^2 u \cdot \nabla_h u_t \, dx + \int \partial_3 u_{3t} \, \nabla_h u \cdot \nabla_h u_t \, dx + \int u_{3t} \, \nabla_h u \cdot \partial_3 \nabla_h u_t \, dx \\ &\le C\|u_t\|_{L^2}^{\frac12} \|\partial_1 u_t\|_{L^2}^{\frac12} \|\nabla_h^2 u\|_{L^2}^{\frac12} \|\partial_2 \nabla_h^2 u\|_{L^2}^{\frac12} \|\partial_3 \nabla_h u_t\|_{L^2} \\ &\quad + C\|\partial_3 u_t\|_{L^2}^{\frac12} \|\partial_1\partial_3 u_t\|_{L^2}^{\frac12} \|\nabla_h u\|_{L^2}^{\frac12} \|\partial_2 \nabla_h u\|_{L^2}^{\frac12} \|\partial_3 \nabla_h u_t\|_{L^2} \\ &\quad + C\|\partial_3 u_t\|_{L^2}^{\frac12} \|\partial_2 u_t\|_{L^2}^{\frac14} \|\partial_2\partial_3 u_t\|_{L^2}^{\frac14} \|\nabla_h u\|_{L^2}^{\frac12} \|\partial_1 \nabla_h u\|_{L^2}^{\frac12} \|\partial_3 \nabla_h u_t\|_{L^2} \\ &\le C\|\nabla_h u\|_{H^{1,0}}^2 \|\partial_3 \nabla_h^2 u\|_{H^{1,0}}^2 \|\nabla u_t\|_{L^2}^2 + \frac{\nu}{36} \|\partial_3 \nabla_h u_t\|_{L^2}^2. \end{aligned}$$

Thus,

$$K_5 \le C\Big(\|\nabla u\|_{L^2} \|u\|_{H^2} + \|\nabla_h u\|_{H^{1,0}}^2\Big) \|\partial_3 \nabla_h u\|_{H^{2,0}}^2 \|\nabla u_t\|_{L^2}^2 + \frac{\nu}{12} \|\partial_3 u_t\|_{H^{1,0}}^2.$$

As in (3.22), $K_6 + K_8$ can be split into six parts as

$$K_6 + K_8 = \int \nabla_h b_t \cdot \nabla b \cdot \nabla_h u_t \, dx + \int b_t \cdot \nabla \nabla_h b \cdot \nabla_h u_t \, dx + \int \nabla_h b \cdot \nabla b_t \cdot \nabla_h u_t \, dx$$
$$+ \int \nabla_h b_t \cdot \nabla u \cdot \nabla_h b_t \, dx + \int b_t \cdot \nabla \nabla_h u \cdot \nabla_h b_t \, dx + \int \nabla_h b \cdot \nabla u_t \cdot \nabla_h b_t \, dx.$$

Similarly to $K_{51}, K_{52}$, and $K_{53}$, we have

$$\int \nabla_h b_t \cdot \nabla b \cdot \nabla_h u_t \, dx + \int \nabla_h b_t \cdot \nabla u \cdot \nabla_h b_t \, dx$$
$$\leq C\|(\nabla u, \nabla b)\|_{L^2} \|(u, b)\|_{H^2} \|(\partial_3 \nabla_h u, \partial_3 \nabla_h b)\|_{H^{2,0}}^2 \|(\nabla u_t, \nabla b_t)\|_{L^2}^2$$
$$+ \left( \frac{\nu}{36} \|\partial_3 \nabla_h u_t\|_{L^2}^2 + \frac{\eta}{24} \|\partial_3 \nabla_h b_t\|_{L^2}^2 \right),$$

$$\int b_t \cdot \nabla \nabla_h b \cdot \nabla_h u_t \, dx + \int b_t \cdot \nabla \nabla_h u \cdot \nabla_h b_t \, dx$$
$$\leq C\|(\nabla_h u, \nabla_h b)\|_{H^{1,0}}^2 \|(\partial_3 \nabla_h^2 u, \partial_3 \nabla_h^2 b)\|_{H^{1,0}}^2 \|\nabla b_t\|_{L^2}^2 + \left( \frac{\nu}{36} \|\partial_3 \nabla_h u_t\|_{L^2}^2 + \frac{\eta}{24} \|\partial_3 \nabla_h b_t\|_{L^2}^2 \right),$$

and

$$\int \nabla_h b \cdot \nabla b_t \cdot \nabla_h u_t \, dx + \int \nabla_h b \cdot \nabla u_t \cdot \nabla_h b_t \, dx$$
$$\leq C\|\nabla_h b\|_{H^{1,0}}^2 \|\partial_3 \nabla_h^2 b\|_{H^{1,0}}^2 \|(\nabla u_t, \nabla b_t)\|_{L^2}^2 + \left( \frac{\nu}{36} \|\partial_3 u_t\|_{H^{1,0}}^2 + \frac{\eta}{24} \|\partial_3 b_t\|_{H^{1,0}}^2 \right).$$

Consequently,

$$K_6 + K_8 \leq C\Big( \|(\nabla u, \nabla b)\|_{L^2} \|(u, b)\|_{H^2} + \|(\nabla_h u, \nabla_h b)\|_{H^{1,0}}^2 \Big)$$
$$\times \|(\partial_3 \nabla_h u, \partial_3 \nabla_h b)\|_{H^{2,0}}^2 \|(\nabla u_t, \nabla b_t)\|_{L^2}^2 + \left( \frac{\nu}{12} \|\partial_3 u_t\|_{H^{1,0}}^2 + \frac{\eta}{8} \|\partial_3 b_t\|_{H^{1,0}}^2 \right).$$

Finally, $K_7$ can also be bounded by

$$K_7 \leq C\Big( \|\nabla b\|_{L^2} \|b\|_{H^2} + \|(\nabla_h u, \nabla_h b)\|_{H^{1,0}}^2 \Big)$$
$$\times \|(\partial_3 \nabla_h u, \partial_3 \nabla_h b)\|_{H^{2,0}}^2 \|(\nabla u_t, \nabla b_t)\|_{L^2}^2 + \left( \frac{\nu}{12} \|\partial_3 u_t\|_{H^{1,0}}^2 + \frac{\eta}{8} \|\partial_3 b_t\|_{H^{1,0}}^2 \right).$$

Collecting all the estimates above for $J_1$ through $J_4$ and $K_1$ through $K_8$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|(u_t,b_t)\|_{H^{1,0}}^2+\nu\|\partial_3 u_t\|_{L^2}^2+\eta\|\partial_3 b_t\|_{L^2}^2\Big)+\Big(\nu\|\partial_3 u_t\|_{H^{1,0}}^2+\eta\|\partial_3 b_t\|_{H^{1,0}}^2+\|(u_{tt},b_{tt})\|_{L^2}^2\Big)\\
&\le C\Big(\|(\nabla u,\nabla b)\|_{L^2}\|(u,b)\|_{H^2}+\|(\partial_3 u,\partial_3 b)\|_{L^2}^2+\|(u,b)\|_{H^{2,0}}^2\Big)\\
&\quad\times\|(\partial_3\nabla_h u,\partial_3\nabla_h b)\|_{H^{2,0}}^2\|(\nabla u_t,\nabla b_t)\|_{L^2}^2\\
&\le C\Big(\|(\partial_3 u,\partial_3 b)\|_{L^2}+\|(u,b)\|_{H^{2,0}}\Big)\|(\partial_3\nabla_h u,\partial_3\nabla_h b)\|_{H^{2,0}}^2\|(\nabla u_t,\nabla b_t)\|_{L^2}^2,
\end{aligned}
$$

where we have used the uniform bound for $\|(u,b)\|_{H^2}$. This completes the proof of Proposition 4.2. ■

Next we prove the decay rate in Theorem 1.1.

**Proof of the decay estimate in Theorem 1.1.** As aforementioned, the proof of the decay estimates is divided into three main steps. The first step shows the exponential decay for $\|(u,b)\|_{H^{3,0}}+\|(\partial_3 u,\partial_3 b)\|_{L^2}$ by making use of the estimate in Proposition 4.1. Adding (3.10) and (4.1), and invoking the global bound in (3.35), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|(u,b)\|_{L^2}^2+\sum_{i=1}^{2}\|(\partial_i^3 u,\partial_i^3 b)\|_{L^2}^2+\nu\|\partial_3 u\|_{L^2}^2+\eta\|\partial_3 b\|_{L^2}^2\Big)\\
&\quad+\big(2c_0\nu\|\partial_3 u\|_{H^{3,0}}^2+2c_0\eta\|\partial_3 b\|_{H^{3,0}}^2+\|(u_t,b_t)\|_{L^2}^2\big)\\
&\le C_0\big(\|(u,b)\|_{H^{3,0}}+\|(u,b)\|_{H^{1,0}}^2\big)\big(\|\partial_3 u\|_{H^{3,0}}^2+\|\partial_3 b\|_{H^{3,0}}^2\big)\\
&\le C_0(\delta+\delta^2)\big(\|\partial_3 u\|_{H^{3,0}}^2+\|\partial_3 b\|_{H^{3,0}}^2\big).
\end{aligned}
$$

If we select $\delta$ to be sufficiently small such that $C_0(\delta+\delta^2)<\min\{2c_0\nu,\ 2c_0\eta\}$, then, for a positive constant $C$,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|(u,b)\|_{L^2}^2+\sum_{i=1}^{2}\|(\partial_i^3 u,\partial_i^3 b)\|_{L^2}^2+\nu\|\partial_3 u\|_{L^2}^2+\eta\|\partial_3 b\|_{L^2}^2\Big)\\
&\quad+C(\|\partial_3 u\|_{H^{3,0}}^2+\|\partial_3 b\|_{H^{3,0}}^2)\le 0.
\end{aligned}
$$

Due to $u|_{\partial\Omega} = b|_{\partial\Omega} = 0$, by virtues of (2.1), we have

$$\|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2} \|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 \le \|u\|_{H^{3,0}}^2 + \|b\|_{H^{3,0}}^2 \le C(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 b\|_{H^{3,0}}^2).$$

Then, for some constant $C_1(\nu,\eta) > 0$,

$$\frac{d}{dt}X(t) + 2C_1 X(t) \le 0,$$

where

$$X(t) = \|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2} \|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \nu\|\partial_3 u\|_{L^2}^2 + \eta\|\partial_3 b\|_{L^2}^2.$$

Therefore,

$$X(t) \le e^{-2C_1 t} X(0)$$

or

$$\|(u,b)\|_{H^{3,0}} + \|(\partial_3 u, \partial_3 b)\|_{L^2} \le Ce^{-C_1 t}(\|(u_0,b_0)\|_{H^{3,0}} + \|(\partial_3 u_0, \partial_3 b_0)\|_{L^2}). \tag{4.3}$$

The second step verifies the exponential decay rate for $\|u_t\|_{H^1} + \|b_t\|_{H^1}$, for any $t > 0$,

$$\|u_t\|_{H^1} + \|b_t\|_{H^1} \le Ce^{-C_3 t}. \tag{4.4}$$

We relies on Proposition 4.2. Adding (3.10) and (4.2), and using (4.3), we deduce that, for a constant $C_2$,

$$\begin{aligned}
&\frac{d}{dt}\Big(\|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2} \|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \|(u_t,b_t)\|_{H^{1,0}}^2 + \nu\|\partial_3 u_t\|_{L^2}^2 + \eta\|\partial_3 b_t\|_{L^2}^2\Big)\\
&\qquad + C(\nu,\eta)(\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2 + \|(\partial_3 u_t, \partial_3 b_t)\|_{H^{1,0}}^2)\\
&\le C\big(\|(\partial_3 u, \partial_3 b)\|_{L^2} + \|(u,b)\|_{H^{3,0}}\big)\|(\partial_3 u, \partial_3 b)\ \|_{H^{3,0}}^2(\|(u_t,b_t)\|_{H^1}^2 + 1)\\
&\le C_2 e^{-C_1 t}\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2,
\end{aligned} \tag{4.5}$$

where we have used the uniform bound of $\|(u_t, b_t)\|_{H^1}^2$. We choose $T > 0$ satisfying

$$C(\nu, \eta, T) := C(\nu, \eta) - C_2 e^{-C_1 T} > 0.$$

Then (4.5) implies that, for $t > T$,

$$\begin{aligned} &\frac{d}{dt}\Big(\|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2}\|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \|(u_t, b_t)\|_{H^{1,0}}^2 + \nu\|\partial_3 u_t\|_{L^2}^2 + \eta\|\partial_3 b_t\|_{L^2}^2\Big) \\ &\quad + C(\nu,\eta,T)(\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2 + \|(\partial_3 u_t, \partial_3 b_t)\|_{H^{1,0}}^2) \le 0. \end{aligned} \tag{4.6}$$

Due to the Poincaré-type inequality (2.1), we have

$$\begin{aligned} &\|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2}\|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 \le \|(u,b)\|_{H^{3,0}}^2 \le C\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2, \\ &\|(u_t, b_t)\|_{H^{1,0}}^2 \le C\|(\partial_3 u_t, \partial_3 b_t)\|_{H^{1,0}}^2 \end{aligned}$$

and thus

$$\begin{aligned} &\|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2}\|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \|(u_t, b_t)\|_{H^{1,0}}^2 + \nu\|\partial_3 u_t\|_{L^2}^2 + \eta\|\partial_3 b_t\|_{L^2}^2 \\ &\quad \le C(\|(\partial_3 u, \partial_3 b)\|_{H^{3,0}}^2 + \|(\partial_3 u_t, \partial_3 b_t)\|_{H^{1,0}}^2). \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), and setting

$$Y(t) = \|(u,b)\|_{L^2}^2 + \sum_{i=1}^{2}\|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \|(u_t, b_t)\|_{H^{1,0}}^2 + \nu\|\partial_3 u_t\|_{L^2}^2 + \eta\|\partial_3 b_t\|_{L^2}^2,$$

we obtain, for a constant $C_3 > 0$,

$$\frac{d}{dt}Y(t) + 2C_3 Y(t) \le 0,$$

which yields

$$Y(t) \le e^{-C_3 t}Y(0).$$

That is, for $t > T$,

$$\|u_t\|_{H^1} + \|b_t\|_{H^1} \leq Ce^{-C_3 t}(\|(u_0, b_0)\|_{H^2}^2 + \|(\nabla_h^3 u_0, \nabla_h^3 b_0)\|_{L^2} + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{H^1}). \tag{4.8}$$

For $0 \leq t \leq T$, by (3.36) and (3.39), it is easy to see that

$$\|u_t\|_{H^1} + \|b_t\|_{H^1} \leq C(\|(u_0, b_0)\|_{H^2}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{H^1}) \leq Ce^{-C_3 t} \tag{4.9}$$

for some constant $C$ depending on the initial data. Then (4.8) and (4.9) give the desired estimate (4.4).

The final step of the proof is to derive the exponential decay rate for $\|(u, b)\|_{H^3}$ using the decay rates from the first two steps, and the Stokes and elliptic regularity estimates. Invoking the estimate (3.37) yields

$$\|\partial_3 \nabla u\|_{L^2} + \|\partial_3 \nabla b\|_{L^2} \leq C(\|(u_t, b_t)\|_{L^2} + \|(\Delta_h u, \Delta_h b)\|_{L^2}) \leq Ce^{-C_4 t}.$$

for a constant $C_4 = \min\{C_1, C_3\}$. Furthermore, according to Proposition 3.5, we have

$$\|\partial_3 \nabla^2 u\|_{L^2} + \|\nabla p\|_{H^1} \leq C(\|u_t\|_{H^1} + \|(u, b)\|_{H^2} + \|\Delta_h u\|_{H^{1,0}}) \leq Ce^{-C_4 t}.$$
$$\|\partial_3 \nabla^2 b\|_{L^2} \leq C(\|b_t\|_{H^1} + \|(u, b)\|_{H^2} + \|\Delta_h b\|_{H^{1,0}}) \leq Ce^{-C_4 t}.$$

This completes the proof of the decay estimate in Theorem 1.1. ■

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