Notes for Lie Theory

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Contents

Chapter 1

Matrix Analysis

1.1 Matrix groups

Last semester, in our reading group on differential topology, we analyzed some common Lie groups by treating them as an embedded submanifold. And we calculated the dimension by implicit function theorem. Now we discuss an algebraic approach to Lie groups. We assume familiarity in group theory, and we consider Lie groups as topological groups, namely, putting a topology on groups. This is going to be used in matrix analysis.

Definition 1.1.1. Let $M(n,k)$ be the set of $n \times n$ matrices, where k is a field (such as $\mathbb{C}, \mathbb{R}$). Let $\mathrm{GL}(n,k) \subset M(n,k)$ be the set of $n \times n$ invertible matrices with entries in k. Then we call this the *general linear group* over k.

Remark 1.1.2. Now we let $k = \mathbb{C}$, and we can consider $M(n, \mathbb{C})$ as $\mathbb{C}^{n^2}$ with the Euclidean topology. Thus we can define basic terms in analysis.

Definition 1.1.3. Let $\{A_m\} \subset M(n,\mathbb{C})$ be a sequence. Then A_m converges to A if each entry converges to the cooresponding entry in A in Euclidean metric space.

Definition 1.1.4. A *matrix Lie group* is a closed subgroup G of $\mathrm{GL}(n,\mathbb{C})$. That is, If A_m is a convergent sequence in G, then A_m converges to some $A \in G$ or A not invertible.

Example 1.1.5. Consider $\mathrm{GL}(n,\mathbb{C})$, it is a matrix Lie group since the either the convergent sequence converges to an invertible matrix or a non-invertible one.

Example 1.1.6. We define $\mathrm{SL}(n,k)$ be the group of $n \times n$ invertivle matrices with entries in k with determinant 1. Then if we have a convergent sequence in $\mathrm{SL}(n,\mathbb{C})$, converging to A, then $\det(A) = 1$. Thus $A \in \mathrm{SL}(n,\mathbb{C})$. Thus $\mathrm{SL}(n,\mathbb{C})$ is a matrix Lie group.

Definition 1.1.7. An $n \times n$ complex matrix A is *unitary* if $A^*A = I$, where A^* is the *adjoint* of A, the complex conjugate of the transpose of A. This is equivalent to $A^* = A^{-1}$

Example 1.1.8. The group of unitary matrices is called the *unitary group*, and we denote it as $\mathrm{U}(n)$. The group of unitary matrices with determinant 1 is called the *special unitary group*, and we denote it as $\mathrm{SU}(n)$. Let $\{A_m\}$ be a convergent sequence in $\mathrm{U}(n)$, converging to A, then $A_m^{-1} \to A^{-1}$ and $A_m^* \to A^*$. And $A_m^* = A_m^{-1}$, Thus $A^* = A^{-1}$ since limit is unique. Thus $\mathrm{U}(n)$ is closed subgroup of $\mathrm{GL}(n, \mathbb{C})$. Thus a matrix group. And $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$. Thus $\mathrm{SU}(n)$ is closed and subgroup of $\mathrm{GL}(n, \mathbb{C})$. Thus a matrix Lie group.

Now, we let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$, by $\langle x, y \rangle = \sum_i \overline{x_i} y_i$. Then we see that the condition on unitary matrix is equivalent to $\langle Ax, Ay \rangle = \langle x, y \rangle$.

Proposition 1.1.9. *If A is unitary,* $|\det(A)| = 1$.

Proof. $\det(A^*A) = |\det(A)|^2 = \det(I) = 1$ since $\det(A^*) = \det(\overline{A})$ and basic complex multiplication. □

Example 1.1.10. An $n \times n$ real matrix A is called *orthogonal* if $A^T = A^{-1}$. Thus it preserves inner product on $\mathbb{R}^n$ as unitary matrices do. And by proposition, $|\det(A)| = \pm 1$. The group of all orthogonal matrices is called the *orthogonal group*, denoted $\mathrm{O}(n)$, and the ones with determinant 1 is called the *special orhtogonal group*, denoted $\mathrm{SO}(n)$. Then $\mathrm{O}(n)$ denotes rotations and reflections and $\mathrm{SO}(n)$ denotes rotations. These are matrix Lie groups, which can be checked as unitary groups.

Now we define a bilinear form $(\cdot, \cdot)$ such that $(x, y) = \sum_i x_i y_i$ (called the dot product), where $x, y \in \mathbb{C}^n$.

Example 1.1.11. The group of $n \times n$ complex valued matrices which preserves the above bilinear form is the *complex orthogonal group*, $\mathrm{O}(n, \mathbb{C})$, which is also a matrix Lie group. Similarly we have another matrix Lie group $\mathrm{SO}(n, \mathbb{C})$.

Example 1.1.12. Let $n, k \in \mathbb{N}$, define $[\cdot, \cdot]_{n,k}$ on $\mathbb{R}^{n+k}$, a symmetric bilinear by $[x, y]_{n,k} = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1} - \ldots - x_{n+k} y_{n+k}$. Then the set of $(n + k) \times (n + k)$ real matrices A such that $[Ax, Ay]_{n,k} = [x, y]$ is called the *generalized orthogonal group*, denoted as $\mathrm{O}(n; k)$ is a matrix Lie group. In physics, one is interested in $\mathrm{O}(3; 1)$, the Lorentz group. We define $\mathrm{SO}(n; k)$ be the subgroup of $\mathrm{O}(n; k)$ with determinant 1.

Example 1.1.13. We define a skew-symmetric bilinear form on $\mathbb{R}^{2n}$ by $\omega(x, y) = \sum_{j=1}^n (x_j y_{n+j} - x_{n+j} y_j)$. The set of all $2n \times 2n$ real matrices A such that $\omega(Ax, Ay) = \omega(x, y)$ is the *real symplectic group* denoted as $\mathrm{Sp}(n, \mathbb{R})$, and this is a matrix Lie group.

Let $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ be a $2n \times 2n$ matrix, then $\omega(x, y) = \langle x, \Omega y \rangle$ by direct computation. Therefore $\omega(Ax, Ay) = \langle Ax, \Omega Ay \rangle = \langle x, A^T \Omega Ay \rangle = \langle x, \Omega y \rangle = \omega(x, y)$ iff $A^T \Omega A = \Omega$. Thus $-\Omega A^T \Omega = A^{-1}$. Thus, one can show that $\det(A)^2 = 1$.

We can define similarly the *Complex Symplectic Group* $\mathrm{Sp}(n, \mathbb{C})$, where all the identity above holds. Finally, the *compact symplectic group* $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n)$. Then this preserves both the bilinear form ω and the inner product we defined above.

Example 1.1.14. The *Euclidean group* $\mathrm{E}(n)$ is the group of all transformation of $\mathbb{R}^n$ that can be expressed by translations and orthogonal linear transformation. Therefore, we can write elements in $\mathrm{E}(n)$ as pairs $\{x, R\}$ with $x \in \mathbb{R}^n$ and $R \in \mathrm{O}(n)$. An we let $\{x, R\}$ act on $\mathbb{R}^n$ by $\{x, R\}y = Ry + x$. Then the operation of this group is given by $\{x_1, R_1\}\{x_2, R_2\}y = R_1(R_2 y + x_2) + x_1 = R_1 R_2 y + x_1 + R_1 x_2 = \{x_1 + Rx_2, R_1 R_2\}y$. And thus, the inverse is given by $\{x, R\}^{-1} = \{-R^{-1}x, R^{-1}\}$. Thus, we observe that an element of $\mathrm{E}(n)$ is isomorphic to the matrix $\begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix}$, $R \in \mathrm{O}(n)$. Therefore, $\mathrm{E}(n)$ is isomorphic to the closed subgroup of $\mathrm{GL}(n+1, \mathbb{R})$.

We define $\mathrm{P}(n, 1)$ the Poincare group (inhomogeneous Lorentz group) be the group of all transformations of $\mathbb{R}^{n+1}$, where an element is in the form $\begin{bmatrix} A & x \\ 0 & 1 \end{bmatrix}$ where $A \in \mathrm{O}(n; 1), x \in \mathbb{R}^{n+1}$.

Example 1.1.15. In this example, we introduce some examples that are more common. The set of all 3×3 real matrices of the form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. This set forms a group called the Heisenberg group. (This is useful in physics as a model for Heisenberg-Weyl commutation relation)

One can associate $\mathbb{R} - \{0\}$ with multiplication to be $\mathrm{GL}(1, \mathbb{R})$, $\mathbb{C} - \{0\}$ with multiplication to be $\mathrm{GL}(1, \mathbb{C})$. And S^1 the sphere, to be $\mathrm{U}(1)$. And if we consider $\mathbb{R}$ with addition, then we can associate it with the set of matrices with positive determinant by the exponential map: $x \mapsto e^x$. Similarly we can do this with $\mathbb{R}^n$.

We end this section with a careful examination of the Compact Symplectic groups by showing that is it the "unitary group over the quaternions". Firstly, define *conjugate-linear* map $J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $J(\alpha, \beta) = (-\bar{\beta}, \bar{\alpha})$ where $\alpha, \beta \in \mathbb{C}^n$ and $(\alpha, \beta) \in \mathbb{C}^{2n}$. Then we observe $J^2 = -I$, and that if $z, w \in \mathbb{C}^{2n}$, then $\omega(z, w) = \langle Jz, w \rangle = -\overline{\langle z, Jw \rangle} = -\langle Jw, z \rangle$.

Proposition 1.1.16. *If $U \in U(2n)$, then $U \in Sp(n)$ if and only if U commutes with J.*

Proof. Let $U \in \mathrm{U}(2n)$. Let $z, w \in \mathbb{C}^{2n}$, then $\omega(Uz, Uw) = \langle JUz, Uw \rangle = \langle U^* JU, w \rangle = \langle U^{-1} JUz, w \rangle$, and we also have $\omega(z, w) = \langle Jz, w \rangle$. Then U preserves ω if and only if $U^{-1} JU = J$. Thus $UJ = JU$. $\square$

One should know the quaternion group $\mathbb{H}$ in group theory, with eight elements such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. In fact the quarternion group can be written in matrix form $\mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Then the quarternion algebra is the space of real linear combinations of $I, \mathbf{i}, \mathbf{j}, \mathbf{k}$.

Since J is conjugate linear, then $J(iz) = -iJ(z)$. Then for $z \in \mathbb{C}^{2n}$, $iJ = -Ji$. Then if we define $K = iJ$, then one can check that iI, J, K satisfy the relations as **i, j, k**.

Therefore, we observe that $U \in \mathrm{Sp}(n)$ commutes with this defined **i,j,k**. Thus, we conclude that $U \in \mathrm{Sp}(n)$ iff U is quaternion linear and preserves the norm. That is why we describe $\mathrm{Sp}(n)$ as the unitary group over the quaternions.

Next we discuss properties of eigenvectors and eigenvalues for $\mathrm{Sp}(n)$.

Lemma 1.1.17. *Let V be a complex subspace of $\mathbb{C}^{2n}$, invariant under J. Then the orthogonal complement $V^\perp$ of V is also invariant under J. Furthermore, $V, V^\perp$ are orthogonal with respect to ω, that is $\omega(z,w) = 0$, for $z \in V, w \in V^\perp$.*

Proof. If $w \in V^\perp$, then $\forall z \in V$, $Jz \in V$, then $\langle Jw, z\rangle = -\langle Jz, w\rangle = \omega(z,w) = 0$, and we know that $V^\perp$ is invariant under J. □

Theorem 1.1.18. *$U \in Sp(n)$, if and only if $\exists u_1, \dots, u_n$ an orthonormal basis, and $v_1, \dots, v_n \in \mathbb{C}^{2n}$ with the following properties: $Ju_i = v_i$ and that $\exists \theta_1, \dots, \theta_n$ such that $Uu_j = e^{i\theta_j}u_j, Uv_j = e^{-i\theta_j}v_j$; finally, $\omega(u_j, u_k) = \omega(v_j, v_k) = 0$, $\omega(u_j, v_k) = \delta_{jk}$, where δ is the Kronecker's delta.*

Proof. Let $U \in \mathrm{Sp}(n)$, choose an eigenvector for U, and normalize it to be a unit vector, say u_1. Then sinder $U \in \mathrm{U}(2n)$, $\lambda_1 = e^{i\theta_1}$ for $\theta \in \mathbb{R}$, and λ_1 is the eigenvalue for u_1. Let $v_1 = Ju_1$, by Prop. 1.1.19, $Uv_1 = J(Uu_1) = J(e^{i\theta_1}u_1) = e^{-i\theta_1}v_1$. Thus v_1 is an eigenvector of U with eigenvalue $e^{-i\theta_1}$. Then $\langle v_1, u_1\rangle = \langle Ju_1, u_1\rangle = \omega(u_1, u_1) = 0$, since ω is skew-symmetric. Moreover, $\omega(u_1, v_1) = \langle Ju_1, v_1\rangle = \langle Ju_1, Ju_1\rangle = 1$.

Then u_1 spanning V is invariant under J and v_1 is invariant under J. By Lemma 1.1.21, one can see that $V^\perp$ is invariant under J and is ω-orthogonal to V. And $V, V^\perp$ is invariant under U and $U*$, the restriction of U to $V^\perp$ gives an eigenvector, and after normailizing it to be a unit one, we call it u_2, and $v_2 = Ju_2$. Then we have the properties we need with u_2, v_2. Then we proceed in this fashion to get the above statement.

Assume the converse. Then let $x, y \in \mathbb{C}^{2n}$ with the basis $u_1, \dots, u_n, v_1, \dots, v_n$. Then by linearity, we can easily check that $\langle Ux, Uy\rangle = \langle x, y\rangle$, and $\omega(Ux, Uy) = \omega(x, y)$. Therefore, $U \in \mathrm{Sp}(n)$. □

1.2 Topological Properties

In this section, we examine some topological properties on matrix Lie groups. And finally we study the topology on projective spaces.

Remark 1.2.1. A matrix Lie group is compact if it is compact in the Euclidean topological sense, when we consider $M(n, \mathbb{C})$ as $\mathbb{R}^{2n}$. Thus the three definitions in a set-point topology/analysis course works. However, since a matrix Lie group is closed in $\mathrm{GL}(n, \mathbb{C})$, we only need the group to be bounded for it to be compact. Thus, we see that $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n)$ are all bounded by 1, since each of their column are unit vectors. However, $\mathrm{SL}(n, \mathbb{R})$ is unbounded, since $A = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \in \mathrm{SL}(n, \mathbb{R})$.

Definition 1.2.2. A matrix Lie group G is *path connected* if $\forall A.B \in G$, $\exists A(t)$ continuous path $a \leq t \leq b$ such that $A(a) = A$, $A(b) = B$. For any matrix Lie group G, the *identity* *component* of G denoted as G_0 is the set $A \in G$ where $\exists A(t)$ continuous path $a \leq t \leq b$ such that $A(a) = I$, $A(b) = A$.

Proposition 1.2.3. *If G is a matrix Lie group, the identity component G_0 of G is a normal subgroup in G.*

Proof. Let $A, B \in G_0$, then $\exists A(t), B(t)$ connecting I to A, B. Then $A(t)B(t)$ is continuous connecting I to AB, and $(A(t))^{-1}$ connecting I to A^{-1}. Thus is a subgroup. Let $A \in G_0$, $B \in G$, then $\exists A(t)$ continuous path connecting I and A. Consider path $BA(t)B^{-1}$. It connects I to BAB^{-1}. Thus $BAB^{-1} \in G_0$, G_0 is normal. □

Proposition 1.2.4. *The group $GL(n, \mathbb{C})$ is path connected for all $n \geq 1$*

Proof. By Spectral theorem, since $A \in \mathrm{GL}(n, \mathbb{C})$ is invertible, $A = STS^{-1}$, where T is upper triangular. Then we let $A(t) = ST(t)S^{-1}$, where we multiply all the entries above the diagonal by $(1 - t)$. Thus, for $0 \leq t \leq 1$, $A(1) = SDS^{-1}$ where D is diagonal. And for $1 \leq t \leq 2$, we consider $A(t) = SD^{-t+1}S^{-1}$, where we finally find a path connecting A and I for al A in $\mathrm{GL}(n, \mathbb{C})$. □

Corollary 1.2.5. *The group $SL(n, \mathbb{C})$ is connected for $n \geq 1$*

Proposition 1.2.6. *$U(n)$ and $SU(n)$ are connected for $n \leq 1$*

Proof. By diagonalization theorem, $A \in \mathrm{U}(n)$ then $A = SDS^{-1}$, where the diagonal are $e^{i\theta_k}$. Then Let $U(t) = SD_nS^{-1}$ where D_n is diagonal with entries $e^{(1-t)i\theta_k}$. Thus $\mathrm{U}(n)$ is connected. Similarly, $\mathrm{SU}(n)$ is connected. □

Proposition 1.2.7. *$SO(n)$ is connected for $n \geq 1$.*

Proof. Let $M \in \mathrm{SO}(n)$. Then $M = ARA^T$, where A is orthogonal and R is a block matrix with rotational matrices as blocks. Then assume the rotation angles are $\theta_1, \ldots, \theta_{n'}$, then let $M(t) = AR(t)A^T$, where $R(t)$ is the matrix replacing the angles θ with $(1 - t)\theta$. Thus $M(t)$ is a path connecting function of I and M. □

The following remarks are fundamentally algebraic topology, an interesting subject on its own. We only look at some properties.

Definition 1.2.8. A matrix Lie group G is simply connected if it is connected and every loop in G can be shrunk continuous to a point in G. I.e. assuming G is connected, then G is simply connected if for every $A(t)$ continuous path, $0 \leq t \leq 1$, in G such that $A(0) = A(1)$, $\exists$ a continuous function $A(s, t)$, where $0 \leq s, t \leq 1$ with the following properties: (1) $A(s, 0) = A(s, 1)$ $\forall s$, (2) $A(0, t) = A(t)$, (3) $A(1, t) = A(1, 0), \forall t$.

Proposition 1.2.9. *$SU(2)$ is simply connected.*

Proof. $\mathrm{SU}(2)$ can be identified with S^3 the three dimensional sphere, which is clearly simply connected. □

We end our section with a careful analysis on the topology of SO(3).

Definition 1.2.10. The *real projective space* of dimension n, denoted as $\mathbb{R}\mathrm{P}^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$.

Remark 1.2.11. Since each line through the origin intersects the unit circle twice, we identify $\mathbb{R}\mathrm{P}^n$ as the unit sphere S^n as pairs $\{u, -u\}$, where $u \in S^n$. The $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ is given by $\pi(u) = \{u, -u\}$. Then we can define a metric on $\mathbb{R}\mathrm{P}^n$ by $d(\{u, -u\}, \{v, -v\} = \min(d(u, v), d(u, -v))$. Thus we see that $\mathbb{R}\mathrm{P}^n$ is locally isometric to S^n (i.e. for nearby points in S^n, the metrics are equal for $\mathbb{R}\mathrm{P}^n$ and S^n).

Moreover, since the upper hemisphere contains the points u or $-u$ for any u, but the equator is identified antipodally. Thus, we can identify $\mathbb{R}\mathrm{P}^n$ as the upper hemisphere with antipodal points on the equator identified. Then we can make a projection into $\mathbb{R}^n$, then we see that the model for $\mathbb{R}\mathrm{P}^n$ is the closed unit disk with antipodal points on the boundary identified.

Remark 1.2.12. We remark that $\mathbb{R}\mathrm{P}^n$ is not simply connected. Assume u is a unit vector in $\mathbb{R}^{n+1}$, $B(t)$ a path connecting $u, -u$. Then $A(t) = \pi(B(t))$ is a closed loop in $\mathbb{R}\mathrm{P}^n$. It suffices to show that this loop can't be contracted to a point. Assume not, then $\exists A(s,t), B(s,t)$ such that $A(s,t) = \pi(B(s,t))$, where $B(0,t) = B(t)$, $A(s,0) = A(s,1)$, then $B(s,0) = \pm B(s,1)$. Since $B(0,0) = -B(0,1)$ for continuity, $B(s,0) = -B(s,1)$. But $0 \notin S^n$. Thus $B(1,t)$ is not a constant function, where $A(1,t)$ is therefore not constant. Thus it is not simply connected.

Proposition 1.2.13. *$SO(3)$ and $\mathbb{R}P^3$ are homeomorphic.*

Proof. Let v be a unit vector in $\mathbb{R}^3$, let $R_{v,\theta}$ be the element of SO(3) with a counterclockwise rotation of the plane orthogonal to v by angle θ. Then we can see that $R_{-v,\theta} = R_{v,\theta}$. Moreover, we show that every element of $SO(3)$ can be expressed as $R_{v,\theta}$, where $-\pi \leq \theta \leq \pi$. Let $R \in \mathrm{SO}(3)$, then R has eigenvalue of norm 1. Since $R$is real, and any nonreal eigenvalues come in conjugate pairs. Thus, there must be a real eigenvalue, namely ± 1. Thus let $v^\perp$ be the plane orthogonal to v, then $Rv = \pm v \in v^\perp$. Thus $Rv = \lambda v$ is a rotation by some angle θ around the axis v.

Let B^3 be the closed ball of radius π in $\mathbb{R}^3$, $\phi : B^3 \to \mathrm{SO}(3)$ by $\phi(u) = R_{\tilde{u},\|u\|}$, where $\tilde{u} = u/\|u\|$, and $\phi(0) = R_{v,0}$. Then ϕ is continuous, since $\theta \to 0, \phi(u) \to 1$. This is surjective since $\|u\| \leq \pi$, and it is injective except for points on the boundary of B^3, which are antipodal, since $R_{v,\pi} = R_{-v,-\pi}$. Since both B^3 and SO(3) are both compact, we know the inverse is also continuous. Thus we see that ϕ is a homeomorphism. Since we can identify B^3 and SO(3) by the above remark, we know that they are homeomorphic. □

Then since $\mathbb{R}\mathrm{P}^3$ is simply connected, SO(3) is simply connected.

1.3 Lie groups and homomorphisms

In this section, we formally introduce the concept Lie groups and some of its properties. We start with recalling objects we studied last semester.

Definition 1.3.1. A *n-dimensional manifold* M is a second-countable, Hausdorff topological space with the property that if $x \in M$ has a neighborhood U being homeomorphic to an open subset of $\mathbb{R}^n$.

We note that this definition is, informally, describing an object locally looks like $\mathbb{R}^n$.

Definition 1.3.2. A *smooth manifold* is a manifold M with a collection of local coordinates covering M such that change of coordinates map between two overlapping coordinate systems is smooth.

This is not going to affect significantly later. But note that one can do calculus on a smooth manifold. Now we introduce the concept of a Lie group.

Definition 1.3.3. A *Lie group* is a smooth manifold G which is also a group, and that the group product $\cdot : G \times G \to G$ and the inverse map $\iota : G \to G$ are smooth.

Example 1.3.4. Let $G = \mathbb{R} \times \mathbb{R} \times S^1 = \{(x, y, u) | x \in \mathbb{R}, y \in \mathbb{R}, u \in S^1 \subset \mathbb{C}\}$ with the group operation $(x_1, y_1, u_1)(x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1y_2}u_1u_2)$. Then G is a group. We quickly check this is true. Firstly, it is clearly a smooth manifold, since the product of smooth manifolds is a smooth manifold. One can easily check that it is closed and that it is associative. Notice that the identity is $(0, 0, 1)$, and $(-x, -y, e^{ixy}u^{-1})$ is clearly an inverse. Thus, one can see that the product and inverse are both smooth. Thus this G is a Lie group.

This example gives a Lie group but it is not a matrix Lie group. This will be proved later. However, all matrix Lie groups are Lie groups by showing that they are embedded submanifolds of $\mathbb{R}^{n^2}$.

Definition 1.3.5. Let G and H be Lie groups, $\phi : G \to H$ is called a *Lie group homomorphism* if (i) ϕ is a group homomorphism (ii) ϕ is smooth. ϕ is called a *Lie group isomorphism* if in addition, it is bijective, and the inverse ϕ^{-1} is smooth.

Remark 1.3.6. In fact, for a matrix Lie group, ϕ is continuous suffices. In this section, we will use this more than the above definition.

Example 1.3.7. Let $\phi : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$, and $\phi(A) = \det(A)$, where $A \in \mathrm{GL}(n, \mathbb{C})$. Then ϕ is a Lie group homomorphism.

Example 1.3.8. Let $\phi : \mathbb{R} \to \mathrm{SO}(2)$ and $\phi(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then ϕ is a Lie group homomorphism

In the following section, we carefully consider the example between SU(2) and SO(3).

Proposition 1.3.9. *There exists a Lie group homomorphism $\phi : SU(2) \to SO(3)$ that is two-to-one and onto.*

Proof. Let V be the space of all 2×2 complex matrices X which are self-adjoint. Then all matrices has the form $\begin{bmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{bmatrix}$. Thus V has dimension 3. Then the standard inner product on $\mathbb{R}^3$ is $\langle X_1, X_2 \rangle = \frac{1}{2}\text{trace}(X_1X_2)$. Let $U \in \text{SU}(2)$, define $\phi_U : V \to V$ by $\phi_U(X) = UXU^{-1}$. We check this is well defined: since U is unitary, $(UXU^{-1})^* = UXU^{-1}$, thus UXU^{-1} is in V.

Now we check it is a homomorphism: $\phi_{U_1U_2}(X) = U_1U_2XU_2^{-1}U_1^{-1} = \phi_{U_1} \circ \phi_{U_2}(X)$. Then we check that it is indeed orthogonal: $\langle \phi_U X_1, \phi_U X_2 \rangle = \frac{1}{2}\text{trace}(UX_1U^{-1}UX_2U^{-1}) = \frac{1}{2}\text{trace}(UX_1X_2U^{-1}) = \frac{1}{2}\text{trace}(X_1X_2) = \langle X_1, X_2 \rangle$. Since SU(2) is connected, ϕ_U must lie in SO(3). Thus $\phi : \text{SU}(2) \to \text{SO}(3)$ by $\phi : U \mapsto \phi_U$ is a homomorphism.

We first consider the kernel of this map. $\phi_U(X) = X$, thus $XU = UX$. We do matrix multiplication and compare the both sides, then conclude that $U = \lambda I$. But since $\det(U) = 1$, $\lambda = \pm 1$. Thus $\ker(\phi) = \{I, -I\}$. Thus ϕ is two to one.

Finally we show that it is onto. Let R be a rotation of V. By Prop 1.2.13, we showed that there is an axis $X \in V$ such that R is a rotation by some angle θ in the pane orthogonal to X. Let $X = U_0 \begin{bmatrix} x_1 & 0 \\ 0 & -x_1 \end{bmatrix} U_0^{-1}$ with $U_0 \in \text{U}(2)$, then the plane orthogonal to X has the form $X_\perp = U_0 \begin{bmatrix} 0 & x_2 + ix_3 \\ x_2 - ix_3 & 0 \end{bmatrix} U_0^{-1}$. We take $U = U_0 \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} U_0^{-1}$. Then $UXU^{-1} = X$, and $UX_\perp U^{-1}$ is the matrix having the form of $X_\perp$ but with x_2, x_3 rotated by some angle θ. Thus ϕ_U is a rotation by angle θ in the plane perpendicular to X. Thus ϕ_U coicides with R. □

1.4 Matrix Exponential

Definition 1.4.1. Let X be an $n \times n$ matrix, the *exponential* of X denoted e^X or $\exp(X)$ is $e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$.

Definition 1.4.2. For $X \in M(n, \mathbb{C})$, let $\|X\| = (\sum_{j,k=1}^{n} |X_{jk}|^2)^{\frac{1}{2}}$. $\|X\|$ is called the *Hilbert-Schmidt norm* of X.

Definition 1.4.3. X_m converges to X iff $\|X_m - X\| \to 0$ as $m \to \infty$.

Remark 1.4.4. We have $\|X + Y\| \leq \|X\| + \|Y\|$, and $\|XY\| \leq \|X\|\|Y\|$. (The first by triangle inequality, the second by Cauchy-Schwarz.

Proposition 1.4.5. *The series* $\sum_{m=0}^{\infty} \frac{X^m}{m!}$ *converges for all* $X \in M(n, \mathbb{C})$ *and* e^X *is a continuous function.*

Proof. By 1.4.4, $\|X^m\| \leq \|X\|^m$. Then we have $|\sum_{m=0}^{\infty} \frac{X^m}{m!}\| \leq \|I\| + \sum_{m=1}^{\infty} \frac{\|X\|^m}{m!} < \infty$. Thus the series converges absolutely.

Since X^m is continuous, the partial sum is continuous. By Weierstrass M-test, the series converges uniformly on each ball of radius R. Thus e^X is continuous. □

Proposition 1.4.6. *Let X, Y be $n \times n$ matrices. Then:*

1. $e^0 = I$
2. $(e^X)^* = e^{X^*}$
3. e^X *is invertible, and* $(e^X)^{-1} = e^{-X}$
4. $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$
5. *if* $XY = YX$, *then* $e^{X+Y} = e^X e^Y = e^Y e^X$ *(note that if* $XY \neq YX$ *it is not necessarily true)*
6. $C \in GL(n, \mathbb{C})$ *then* $e^{CXC^{-1}} = Ce^X C^{-1}$.

Proof. For 1, By definition, $e^0 = I + \sum 0 = I$. For 2, $(e^X)^* = (\sum_{m=0}^{\infty} \frac{X^m}{m!})^* = \sum_{m=0}^{\infty} (\frac{X^m}{m!})^* = \sum_{m=0}^{\infty} \frac{(X^*)^m}{m!} = e^{X^*}$. Assuming 5, we know that $XX^{-1} = X^{-1}X$. Thus $e^X (e^X)^{-1} = e^{X+-X} = I$. And for 4, $\alpha X \beta X = \beta X \alpha X$. Thus is true. Therefore, it suffices to prove 5. We do it by direct computation. $e^X e^Y = \sum_{m=0}^{\infty} \frac{X^m}{m!} \sum_{m=0}^{\infty} \frac{Y^m}{m!} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{X^k Y^{m-k}}{k!(m-k)!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k Y^{m-k} = \sum_{m=0}^{\infty} \frac{1}{m!} (X+Y)^m = e^{X+Y}$. Finally, on 6, $e^{CXC^{-1}} = \sum_{m=0}^{\infty} \frac{(CXC^{-1})^m}{m!} = \sum_{m=0}^{\infty} \frac{CX^m C^{-1}}{m!} = Ce^X C^{-1}$. □

Proposition 1.4.7. *Let X be a $n \times n$. Then e^{iX} is a smooth curve in $M(n, \mathbb{C})$ and* $\frac{d}{dt} e^{tX} = Xe^{tX} = e^{tX} X$ *and* $\frac{d}{dt} e^{tX}|_{t=0} = X$. *(Note that* $\frac{d}{dt} e^{X+tY}$ *might not be* Ye^{X+tY}*)*

Proof. We differentiate term by term since the series is absolutely convergent proved in Prop 1.4.5. □

Now we review some linear algebra in order to compute the exponential. By the spectral theorem, if $X \in M(n, \mathbb{C})$ having n linearly independent eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, and let C be matrix with columns of the eigenvetors, D be diagonal with entries being eigenvalues. Then $X = CDC^{-1}$. Then $e^X = Ce^D C^{-1}$, where e^D is diagonal with entries $e^{\lambda_1}, \ldots, e^{\lambda_n}$.

Another way to decompose it is that, if X is nilpotent, then e^X vanishes somewhere. Let $X = S + N$, where S is diagonalizable, N nilpotnet and $SN = NS$, then $e^X = e^S e^N$.

Remark 1.4.8. Matrix exponential can be used in differential equations. Consider $\frac{dv}{dt} = Xv$ and $v(0) = v_0$, where $v(t) \in \mathbb{R}^n$ and X is an $n \times n$ matrix. Then the solution is $v(t) = e^{tX} v_0$.

1.5 Matrix Logarithm

We define matrix logarithm in this section.

Definition 1.5.1. Let A be an $n \times n$ matrix. Then $\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$ when the series converges.

Theorem 1.5.2. $\log A$ *(in the above definition) is defined and continuous on the set of all $n \times n$ complex matrices with $\|A - I\| < 1$, and under this condition, $e^{\log A} = A$. For $X \in M(n, \mathbb{C})$ with $\|X\| < \log 2$, we have $\|e^X - I\| < 1$ and $\log e^X = X$.*

Proof. Since $\|(A-I)^m\| \leq \|A-I\|^m$, and it has radius of convergence 1, the series converges absolutely for all A where $\|A-I\| < 1$. And continuity follows similarly to the case of matrix exponential.

Assume A satisfies $\|A - I\| < 1$, If A is diagonalizable with eigenvalues $z_1, \dots, z_n$, then $A = CDC^{-1}$ where D is a diagonal matrix, with $(A - I)^m = CD_1C^{-1}$, where D_1 has entries $(z_i - 1)^m$. Let $X \in M(n, \mathbb{C})$, then $\|X\|^2 = \sum_{j,k=1}^n |X_{jk}|^2 = \sum_{j,k=1}^n |\langle e_j, Xe_k\rangle|^2$. Therefore, we see that if v is an eigenvector for X with eigenvalue λ, $\|X\|^2 = \sum |\lambda|^2$. Thus $|\lambda| \leq \|X\|$. Thus with $\|A - I\| < 1$ each eigenvalue z_j of A must satisfy $|z_j - 1| < 1$, thus by summing the series $\log A = CD_2C^{-1}$ where the diagonal of D_2 has the form $\log z_i$. Therefore $e^{\log A} = CD_3C^{-1}$ where D_3 has diagonal $e^{\log z_i} = z_i$ in this case.

Assume then A is not diagonalizable, let A_m be a sequence of diagonalizable matrices, which converges to A. (This we do by considering the eigenvalues of A, say $\lambda_1, \dots \lambda_n$, we make the eigenvalues of A_m to be $\lambda_1 + \frac{1}{m+1}, \dots, \lambda_n + \frac{1}{m+n}$, then the eigenvalues are clearly distinct for most m. Thus A_m are diagonalizable, and their eigenvalues, as $m \to \infty$, converges to the eigenvalues of A. Thus $A_m \to A$.) Then taking the limit, by continuity of logarithm and exponential functions $\exp(\log A) = A$ for A with $\|A - I\| < 1$.

Assume $\|X\| < \log 2$, then $\|e^X - I\| = \|\sum_{m=0}^\infty \frac{X^m}{m!} - I\| \leq \sum_{m=0}^\infty \frac{\|X\|^m}{m!} - 1 = e^{\|X\|} - 1 < 1$. Thus $\log(\exp X)$ makes sense. Thus $\log e^X = X$. □

Proposition 1.5.3. *There exists a constant c such that for all $n \times n$ matrices B with $\|B\| < \frac{1}{2}$, we have $\|\log(I + B) - B\| \leq c\|B\|^2$. In other words, $\log(I + B) = B + O(\|B\|^2)$.*

Proof.

$$\begin{aligned} \|\log(I + B) - B\| &= \|\sum_{m=2}^\infty (-1)^{m+1} \frac{B^m}{m}\| = \|B^2 \sum_{m=2}^\infty (-1)^{m+1} \frac{B^{m-2}}{m}\| \\ &\leq \|B\|^2 \sum_{m=2}^\infty \frac{(\frac{1}{2})^{m-2}}{m} \end{aligned}$$

□

Theorem 1.5.4. *Every invertible $n \times n$ matrix can be expressed as e^X for some $X \in M(n, \mathbb{C})$.*

Proof. Consider a Jordan decomposition of A where $A = S(J_{k_1}(\lambda_1) \oplus \dots \oplus J_{k_m}(\lambda_m))S^{-1}$, where none of the eigenvalues are 0. Then $\lambda_i = e^{z_i}$ for some $z_i \in \mathbb{C}$. Then consider $Y_i = J_{k_i}(z_i)$, then $\exp(Y_i)$ is an upper triangular matrix similar to $J_{k_i}(\lambda_i)$. Then $J_{k_i}(\lambda_i) = P_i \exp(Y_i) P_i^{-1}$ where P_i is invertible. Then define $X_i = P_i Y_i P_i^{-1}$. Therefore, we see that $A = \exp(S(X_1 \oplus \dots \oplus X_m)S^{-1})$. □

1.6 Polar Decomposition and More

We first develop more results on exponentials before we consider polar decomposition. These results will be helpful later.

Theorem 1.6.1 (Lie Product Formula)**.** *For $X, Y \in M(n, \mathbb{C})$, then $e^{X+Y} = \lim_{m\to\infty}(e^{\frac{X}{m}}e^{\frac{Y}{m}})^m$*

Proof. By direct computation, we have $e^{\frac{X}{m}}e^{\frac{Y}{m}} = I + \frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2})$, then $m \to \infty$, we have $e^{\frac{X}{m}}e^{\frac{Y}{m}} \to I$. Therefore, it is in the domain of the log for large m. By Proposition 1.5.3,

$$\begin{aligned}
\log(e^{\frac{X}{m}}e^{\frac{Y}{m}}) &= \log(I + \frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2})) \\
&= \frac{X}{m} + \frac{Y}{m} + O(\|\frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2}\|^2) \\
&= \frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2})
\end{aligned}$$

Therefore, $e^{\frac{X}{m}}e^{\frac{Y}{m}} = \exp(\frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2}))$ and that $(e^{\frac{X}{m}}e^{\frac{Y}{m}})^m = \exp(X + Y + O(\frac{1}{m}))$. Finally, by the continuity of the exponential, we have $e^{X+Y} = \lim_{m\to\infty}(e^{\frac{X}{m}}e^{\frac{Y}{m}})^m$. □

Theorem 1.6.2. *Let $X \in M(n, \mathbb{C})$, we have* $\det(e^X) = e^{tr(X)}$

Proof. If X is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$, then e^X is diagonalizable with eigenvalues $e^{\lambda_1}, \dots, e^{\lambda_n}$. Then $\det(e^X) = e^{\lambda_1} \dots e^{\lambda_n} = e^{\lambda_1 + \dots + \lambda_n} = e^{\text{tr}(X)}$. If X is not diagonalizable, in the proof of Theorem 1.5.2, we can approximate X by $\{X_m\}$ a sequence of diagonalizable matrix, and then we get the same result. □

Definition 1.6.3. A function $A : \mathbb{R} \to \text{GL}(n, \mathbb{C})$ is called a *one-parameter subgroup* of $\text{GL}(n, \mathbb{C})$ if (1) A is continuous, (2) $A(0) = I$, (3) $A(t + s) = A(t)A(s)$ for $t, s \in \mathbb{R}$.

Lemma 1.6.4. *Fix $\varepsilon < \log 2$, let $B_{\frac{\varepsilon}{2}}$ be the ball of radius $\frac{\varepsilon}{2}$ around the origin in $M(n, \mathbb{C})$, let $U = \exp(B_{\frac{\varepsilon}{2}})$. Then every $B \in U$ has a unique square root C in U with $C = \exp(\frac{1}{2}\log B)$.*

Proof. The above C clearly is a square root and in U. We show the uniqueness. Let $C' \in U$ with $(C')^2 = B$, let $Y = \log C'$, then $e^Y = C'$. Thus $e^{2Y} = (C')^2 = B = \exp(\log B)$. Therefore, sicne $Y \in B_{\frac{\varepsilon}{2}}, 2Y \in B_\varepsilon$. Then $\log B \in B_\varepsilon$. By Theorem 1.5.2, exp is injective in B_ε. Thus $C' = C$. □

Theorem 1.6.5. *If $A(\cdot)$ is a one-parameter subgroup of $GL(n, \mathbb{C})$, then $\exists!$ $n \times n$ complex matrix X such that $A(t) = e^{tX}$.*

Proof. For uniqueness, we consider $X = \frac{d}{dt}A(t)|_{t=0}$. Then clearly if there is one, then it is unique. For existence, let U be that in the lemma, since A is continuous, $\exists t_0 > 0$ such that $A(t) \in U$ for all $|t| \le t_0$. Let $X = \frac{1}{t_0}\log(A(t_0))$, then $t_0 X = \log(A(t_0))$. Then $t_0 X \in B_{\frac{\varepsilon}{2}}$ and $e^{t_0 X} = A(t_0)$. Then $A(\frac{t_0}{2}) \in U$, and $A(\frac{t_0}{2}) = A(t_0)$. Thus by lemma,

$A(\frac{t_0}{2}) = \exp(\frac{t_0 X}{2})$, then applying repeatedly, we have $A(\frac{t_0}{2^k}) = \exp(\frac{t_0 X}{2^k})$. Then for any integer m, $A(\frac{mt_0}{2^k} = A(\frac{t_0}{2^k})^m = \exp(\frac{mt_0 X}{2^k})$. Thus $A(t) = \exp tX$ for numbers $\frac{mt_0}{2^k}$, which is dense in $\mathbb{R}$. Thus $A(t) = \exp(tX)$ for all real numbers t. □

Proposition 1.6.6. *The exponential map is an infinitely differentiable map of $M(n, \mathbb{C})$ into $M(n, \mathbb{C})$.*

Proof. For any j, k, $(X^m)_{jk}$ is a homogeneous polynomial of degree m in the entries, thus it is a convergent power series in $M(n, \mathbb{C}) \cong \mathbb{R}^{2n^2}$. Thus we can differentiate term by term. □

Now we start doing polar decomposition. We explain the intuition of polar decomposition. For example consider a complex number z, then $z = re^{i\theta}$, where r is positive real number and $|e^{i\theta}| = 1$. Since r is positive, $r = e^x$ for some $x \in \mathbb{R}$. Thus $z = up$ where $|u| = 1, p$ is positive and real. Then we establish a similar result in matrices.

Definition 1.6.7. P a self-adjoint $n \times n$ matrix ($P^* = P$) is positive if $\langle v, Pv \rangle > 0$ for all $v \in \mathbb{C}^n$.

Lemma 1.6.8. *If Q is a self-adjoint positive matrix, then Q has a unique positive self-adjoint square root.*

Proof. by linear algebra, Q is self-adjoint, then Q has an orthonormal basis of eigenvectors, then $Q = UDU^{-1}$ where D is diagonal with entries positive eigenvalues. Then $Q^{\frac{1}{2}} = U\sqrt{D}U^{-1}$, thus is still self-adjoint and positive.

Then we show it is unique. If $\sqrt{Q}$ is self-adjoint and positive, then Q has the same eigenspace as $\sqrt{Q}$, and the eigenvalue is the square of that of $\sqrt{Q}$. Since $x \mapsto x^2$ is injective, distinct eigenvectors corresponds to distinct eigenvalues. Thus $\sqrt{Q}$ is uniquely determined by Q. □

Theorem 1.6.9.

1. *if $A \in GL(n, \mathbb{C})$ can be written uniquely in the form $A = UP$ where I is unitary and P is self-adjoint and positive.*
2. *Every self-adjoint positive matrix P can be written uniquely in the form $P = e^X$, where X is self adjoint. Conversely, if X is self-adjoint, then e^X is self-adjoint and positive.*
3. *Let $A \in GL(n, \mathbb{C})$, then $A = Ue^X$ where U unitary, X self-adjoint, then U, X depend continuously on A*

Proof. For existence in 1, for any A, A^*A is clearly self-adjoint. If let A be invertible, let $v \in \mathbb{C}^n$, then $\langle v, A^*Av \rangle = \langle Av, Av \rangle > 0$. Then A^*A is positive. We define $P = \sqrt{A^*A}$, and we define $U = AP^{-1} = A\sqrt{A^*A}^{-1}$. Then we only need to check U is unitary. $U^*U = \sqrt{A^*A}^{-1} A^*A \sqrt{A^*A}^{-1} = I$. Thus we have constructed the equation we desired. For uniqueness of 1, if $A = UP$, then $A^*A = PU^*UP = P^2$. Then by Lemma, we see that P is unique. Thus U is unique.

2 can be proved in the same fashion as our lemma but change the square root function to a logarithm function.

Then we prove 3. We claim that $\log P$, where P self-adjoint and positive, depends continuously on P. Assume the eigenvalues of P are between 0 and 2, then by Theorem 1.5.2, $\log P$ converges, and continuity follows. In general, fix some positive, self-adjoint matrix P_0 and a small neighborhood V of P_0 and pick $a > 0$. For $P \in V$, write $P = e^a(e^{-a}P)$. Then the unique self-adjoint $\log P$ can be computed as $\log P = aI + \log(e^{-a}P)$. If a is large enough, then $\forall P \in V$, the eigenvalues of e^{-a} will be less than 2. Then $\log(e^{-a}P)$ converges and depend continuously on P. □

Finally, we look at some applications of the theorem in $\mathrm{GL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{R})$.

Proposition 1.6.10. *1. Every $A \in GL(n, \mathbb{R})$ can be written uniquely as $A = Re^X$, where $R \in O(n)$ and X is real and symmetric.*

2. Every $A \in SL(n, \mathbb{C})$ can be written uniquely as $A = Ue^X$, where $R \in SU(n)$ and X is self-adjoint and with $tr(X) = 0$.

3. Every $A \in SL(n, \mathbb{R})$ can be written uniquely as $A = Re^X$, where $R \in SO(n)$ and X is real and symmetric and $tr(X) = 0$.

Proof. If A is real, then $A^*A = A^TA$ is real and symmetric. Thus $P = \sqrt{A^TA}$ is real, and $U = AP^{-1}$ is real and unitary. Thus in $O(n)$.

If $A \in \mathrm{SL}(n, \mathbb{C})$, then $A = Ue^X$ with $U \in \mathrm{U}(n)$ and X self-adjoint, then $\det(A) = \det(U)e^{\mathrm{tr}(X)} = 1$, then $\mathrm{tr}(X) = 0$, and is unique by polar decomposition.

And if $A \in \mathrm{SL}(n, \mathbb{R})$, then by combining the previous result, $A = Re^X$ where $R \in \mathrm{SO}(n)$, and X is real and symmetric and its trace is zero. □

Chapter 2

Lie Algebra and Representation

2.1 Definitions

Definition 2.1.1. A *finite-dimensional real or complex Lie algebra* is a finite-dimensional real or complex vector space $\mathfrak{g}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying $[X,Y] = -[Y,X]$ for $X,Y \in \mathfrak{g}$ (skew symmetric), and $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for $X,Y,Z \in \mathfrak{g}$ (Jacobi identity).

This binary map is called the *bracket/commutator* on $\mathfrak{g}$ or the *Lie bracket* of $\mathfrak{g}$. X,Y commute if $[X,Y] = 0$ and $\mathfrak{g}$ is commutative if $[X,Y] = 0$ for all $X,Y \in \mathfrak{g}$.

Example 2.1.2. Let $\mathfrak{g} = \mathbb{R}^3$, $x,y \in \mathfrak{g}$ and $[x,y] = x \times y$, where $x \times y$ is the cross product. Then $\mathfrak{g}$ is a Lie algebra.

Example 2.1.3. Let $\mathcal{A}$ be an associative algebra, $\mathfrak{g}$ be a subspace of $\mathcal{A}$ with $XY - YX \in \mathfrak{g}$ for $X,Y \in \mathfrak{g}$. Then $\mathfrak{g}$ is a Lie algebra with bracket $[X,Y] = XY - YX$.

Example 2.1.4. Let $\mathfrak{sl}(n,bC)$ be the space of all $X \in M(n,\mathbb{C})$ for which $\mathrm{tr}(X) = 0$. Then $\mathfrak{sl}(n,\mathbb{C})$ is a Lie algebra with bracket $[X,Y] = XY - YX$.

Definition 2.1.5. A *subalgebra* of a real and complex Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$, such that $[H_1,H_2] \in \mathfrak{h}$ for all $H_1,H_2 \in \mathfrak{h}$. If $\mathfrak{g}$ is complex and $\mathfrak{h}$ is real, then $\mathfrak{h}$ is the *real subalgebra* of $\mathfrak{g}$.

Definition 2.1.6. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* in $\mathfrak{g}$ if $[X,H] \in \mathfrak{h}$ for $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$.

Definition 2.1.7. The *center* of a Lie algebra $\mathfrak{g}$ is the set of all $X \in \mathfrak{g}$ with $[X,Y] = 0$ for all $Y \in \mathfrak{g}$

Definition 2.1.8. If $\mathfrak{g}$, $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if $\phi([X,Y]) \to [\phi(X),\phi(Y)]$ for $X,Y \in \mathfrak{g}$. If ϕ is one-to-one and onto, then ϕ is a *Lie algebra isomorphism.* A Lie algebra isomorphism of a Lie algebra with itself if called a *Lie algebra automorphism.*

Definition 2.1.9. If $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, define a linear map $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by $\text{ad}_X(Y) = [X, Y]$. The map $X \mapsto \text{ad}_X$ is the *adjoint map* or *adjoint representation*

With the above notation, we see that the Jacobi identity can be written as $\text{ad}_X[Y, Z] = [\text{ad}_X(Y), Z] + [Y, \text{ad}_X(Z)]$. Then we observe that ad_X looks like a *derivation* of the bracket (recall the product rule in derivatives)

Proposition 2.1.10. *If $\mathfrak{g}$ is a Lie algebra, then $ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X = [ad_X, ad_Y]$, that is, $ad : \mathfrak{g} \to End(\mathfrak{g})$ is a Lie algebra homomorphism.*

Proof. Since $\text{ad}_{[X,Y]}(Z) = [[X, Y], Z]$, and $[\text{ad}_X, \text{ad}_Y](Z) = [X, [Y, Z]] - [Y, [X, Z]]$ which is the Jacobi Identity. □

Definition 2.1.11. If $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras, the *direct sum* of $\mathfrak{g}_1, \mathfrak{g}_2$ is the vector space direct sum of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, with bracket given by $[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2])$. If $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g}_1, \mathfrak{g}_2$ are subalgebras, $\mathfrak{g}$ *decomposes as the Lie algebra of direct sum* of $\mathfrak{g}_1, \mathfrak{g}_2$ if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and $[X_1, X_2] = 0$ for all $X_1 \in \mathfrak{g}_1, X_2 \in \mathfrak{g}_2$.

Definition 2.1.12. let $\mathfrak{g}$ be a Lie algebra, let $X_1, \ldots, X_N$ be a basis for $\mathfrak{g}$, then the unique constants c_{jkl} such that $[X_j, X_k] = \sum_{l=1}^{N} c_{jkl} X_l$ are called the *structure constants* of $\mathfrak{g}$. (This shows up more in physics)

Then we look at specific types of Lie algebra

Definition 2.1.13. A Lie algebra $\mathfrak{g}$ is *irreducible* if the only ideals are $\{0\}$ and $\mathfrak{g}$. $\mathfrak{g}$ is *simple* if it is irreducible and $\dim \mathfrak{g} \geq 2$.

Remark 2.1.14. One dimensional Lie algebras are special. They are irreducible and they are commutative. Moreover, if a Lie algebra is commutative and irreducible, then it is one dimensional, since if it is commutative, any subspace is an ideal.

Proposition 2.1.15. *The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is simple.*

Proof. Since this is the space of matrices with trace being 0, then its basis is $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We do come computation to see $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$. Let $\mathfrak{h}$ be an ideal in $\mathfrak{sl}(2, \mathbb{C})$, and $\mathfrak{h}$ contains $Z = aX + bH + cY$, where $Z \neq 0$. Then it suffices to show that $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C})$. Then assume $c \neq 0$, $[X, [X, Z]] = [X, [-2bX + cH]] = -2cX$, which is not 0. Since $\mathfrak{h}$ is an ideal, $X \in \mathfrak{h}$. and $[Y, X] = H$ which is non zero, and $[Y, [Y, X]]$ is a nonzero multiple of Y. Thus $Y, H \in \mathfrak{h}$, $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C})$. Then we assume $c = 0, b \neq 0$, and do the same argument, and assume $c = 0, b = 0, a \neq 0$ and do the same argument to get the proposition. □

Definition 2.1.16. If $\mathfrak{g}$ is a Lie algebra, then the *commutator ideal* in $\mathfrak{g}$ denoted $[\mathfrak{g}, \mathfrak{g}] = \{Z \in \mathfrak{g} | Z = c_1[X_1, Y_1] + \ldots + c_m[X_m, Y_m], c_j \text{ is a constant, } X_j, Y_j \in \mathfrak{g}\}$

Definition 2.1.17. Let $\mathfrak{g}$ be a Lie algebra, define sequences of subalgebras $\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots$ of $\mathfrak{g}$ by $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$ etc.. This is called the derived series of $\mathfrak{g}$. A Lie algebra is called *solvable* if $\mathfrak{g}_j = \{0\}$ for some $\jmath$.

Definition 2.1.18. Let $\mathfrak{g}$ be a Lie algebra, define sequences of subalgebras $\mathfrak{g}^j$ by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{j+1} = \{[X, Y] | X \in \mathfrak{g}, Y \in \mathfrak{g}^j\}$. This is called the *upper central series of* $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{g}^j = \{0\}$ for some j.

Proposition 2.1.19. *If $\mathfrak{g} \in M(3, \mathbb{R})$ be the space of 3×3 upper triangular matrices with zeros on the diagonal. Then $\mathfrak{g}$ is a nilpotent Lie algebra.*

Proof. We consider the basis X where $X_{12} = 1$ others 0, Y where $Y_{23} = 1$ others 0, and $Z_{13} = 1$ others 0. Then by computation, $[X, Y] = Z, [X, Z] = [Y, Z] = 0$. Then $[\mathfrak{g}, \mathfrak{g}]$ is the span of Z, and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. Thus nilpotent. □

Proposition 2.1.20. *If $\mathfrak{g} \in M(2, \mathbb{C})$ be the set of matrices $\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in \mathbb{C}\}$, then $\mathfrak{g}$ is solvable but not nilpotent.*

Proof. We compute $[X, Y]$, $X, Y \in \mathfrak{g}$ and see that the commutator ideal is one dimensional. Therefore, $\mathfrak{g}$ is solvable. But it is not nilpotent. $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $[H, X] = 2X$. Thus we see that is upper central series is nonzero multiple of X. Thus $\mathfrak{g}^j \neq \{0\}$ for all j □

2.2 Lie algebra of Matrix Group

Definition 2.2.1. Let G be a matrix Lie group. The *Lie algebra of* G, denoted $\mathfrak{g} = \{X | \forall t \in \mathbb{R}, e^{tX} \in G\}$, where X are matrices.

Remark 2.2.2. We have a equivalent statement, $X \in \mathfrak{g}$ iff the entire one-parameter subgroup generated by X is in G. And that in fact, $\mathfrak{g}$ is the tangent space to G at the identity.

Proposition 2.2.3. *Let G be a matrix Lie group, X an element of its Lie algebra. Then e^X is an element of the identity component G_0 of G.*

Proof. Since $e^{tX} \in G$ for all $t \in \mathbb{R}$. However, as t goes from 0 to 1, e^{tX} is a continuous path connecting the identity to e^X. □

Theorem 2.2.4. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. If X and Y are elements of $\mathfrak{g}$ then (i) $AXA^{-1} \in \mathfrak{g}$ for all $A \in G$. (ii) $sX \in \mathfrak{g}$ for all real numbers s. (iii) $X + Y \in \mathfrak{g}$. (iv) $XY - YX \in \mathfrak{g}$.*

Proof. (i) $e^{tAXA^{-1}} = Ae^{tX}A^{-1} \in G$, therefore, $AXA^{-1} \in \mathfrak{g}$. (ii) Clearly, $e^{t(sX)} = e^{stX} \in G$. Then $sX \in \mathfrak{g}$. (iii) By Lie product formula, $e^{t(X+Y)} = \lim_{m\to\infty}(e^{tX/m}e^{tY/m})^m$. Since

$e^{tX/m}e^{tY/m})^m$ is in G, and G is closed, then $X+Y \in \mathfrak{g}$. (iv) $\frac{d}{dt}(e^{tX}Ye^{tX})|_{t=0} = (XY)e^0 + (e^0Y)(-X) = XY - YX$. Then by (i)$e^{tX}Ye^{tX} \in G$ for all t, by (ii), (iii), $\mathfrak{g}$ is a real subspace of $M(n,\mathbb{C})$ and $\mathfrak{g}$ is closed in $M(n,\mathbb{C})$. Thus $XY - YX = \lim_{h\to 0} \frac{e^{tX}Ye^{tX}-Y}{h} \in \mathfrak{g}$. □

Definition 2.2.5. A matrix Lie group G is complex if its Lie algebra $\mathfrak{g}$ is a complex subspace of $M(n,\mathbb{C})$, where $iX \in \mathfrak{g}$ if $X \in \mathfrak{g}$.

Proposition 2.2.6. *If G is commutative, then $\mathfrak{g}$ is commutative.*

Proof. For $X, Y \in M(n,\mathbb{C})$, the commutator $[X,Y] = \frac{d}{dt}(\frac{d}{ds}e^{tX}e^{sX}e^{-tX}|_{s=0})|_{t=0}$. If G is commutative, $X, Y \in \mathfrak{g}$, then e^{tX} commutes with e^{sY} and thus the right sides is independent of t so $[X,Y] = 0$. □

Now we look at examples of Lie algebras

Proposition 2.2.7. *The Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ of $GL(n,\mathbb{C})$ is the space $M(n,\mathbb{C})$. The Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ is equal to $M(n,\mathbb{R})$. The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ of $SL(n,\mathbb{C})$ consists of all $n\times n$ complex matrices with trace zero. The Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ of $SL(n,\mathbb{R})$ consists of all $n\times n$ real matrices with trace zero.*

Proof. If $X \in M(n,\mathbb{C})$, then $e^{tX} \in \mathrm{GL}(n,\mathbb{C})$ is invertible, so $X \in \mathfrak{gl}(n,\mathbb{C})$. And if similarly we have $\mathfrak{gl}(n,\mathbb{R}) \supset M(n,\mathbb{R})$. Conversely, if e^{tX} is real, then $X = \frac{d(e^{tX})}{dt}|_{t=0}$ is also real. Thus $\mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$. If $X \in M(n,\mathbb{C})$ with $\mathrm{tr}(X) = 0$. Then , $\det(e^{tX}) = e^{\mathrm{tr}(tX)} = 1$. Then $X \in \mathfrak{sl}(n,\mathbb{C})$. Conversely, if $\det(e^{tX}) = e^{\mathrm{tr}(tX)} = 1$, then $\mathrm{tr}(X) = \frac{d}{dt}e^{\mathrm{tr}(X)t}|_{t=0}$. Finally, if X is real $\mathrm{tr}(X) = 0$, then e^{tX} is real with determinant 1. Thus $X \in \mathfrak{sl}(n,\mathbb{R})$. Conversely, it follows the same way as the preceding argument. □

Proposition 2.2.8. *The Lie algebra $\mathfrak{u}(n)$ of $U(n)$ consists of all complex matrices satisfying $X^* = -X$ and the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ consists of all complex matrices satisfying $X^* = -X$ and $tr(X) = 0$. The Lie algebra $\mathfrak{so}(n)$ of $O(n)$ consists of all real matrices X satisfying $X^{tr} = -X$ and the Lie algebra of $SO(n)$ is the same as that of $O(n)$.*

Proof. A matrix e^{tX} is unitary iff $(e^{tX})^* = (e^{tX})^{-1} = e^{-tX}$. And since $(e^{tX})^* = e^{tX^*}$. Therefore, $e^{tX^*} = e^{-tX}$. This holds for all t iff $X^* = -X$, which is exactly $\mathfrak{u}(n)$. And adding the $\det(X) = 1$ condition at lie group level, we have $\mathrm{tr}(X) = 0$ conditino at Lie algebra level. The same argument can be done over $\mathbb{R}$ to show $\mathfrak{so}(n)$ is the Lie algebra of $\mathrm{O}(n)$. And since $X^{\mathrm{tr}} = -X$, $\mathrm{tr}(X) = 0$ naturally. Therefore the Lie algebras of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are the same. □

Proposition 2.2.9. *Let $g = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}$, then the Lie algebra $\mathfrak{so}(n,k)$ of $O(n,k)$ consists precisely of the real matrices X with $gX^Tg = -X$. And the Lie algebra of $SO(n,k)$ is the same as that of $O(n,k)$. If $\Omega = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, then the Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ consists real matrices X with $\Omega X^T \Omega = X$. And the Lie algebra $Sp(n,\mathbb{C})$ has the complex X satisfying* *the same condition. The Lie algebra $\mathfrak{sp}(n)$ consists the complex matrices with $\Omega X^T \Omega = X$ and $X^* = -X$.*

The proof follows the same computation as the above one

Problem 2.2.10. Prove the above proposition.

Proposition 2.2.11. *The Lie algebra of the Heisenberg group H is the space of all matrices of the form $X = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, where $a, b, c \in \mathbb{R}$.*

Proof. If X is strictly upper triangular, then X^m will be strictly upper triangular. Thus for X, $e^{tX} = I + B$ with B strictly upper triangular. Thus $e^{tX} \in H$. Conversely, if $e^{tX} \in H$ for real t, then all the entries are on and below the diagonal are independent of t Then $X = \frac{d(e^{tX})}{dt}|_{t=0}$ is in the form of X. □

Problem 2.2.12. Find the Lie algebras of the Euclidean group and the Poincare group.

Example 2.2.13. The basis for the Lie algebra $\mathfrak{su}(2)$ are $E_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $E_2 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $E_3 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, with commutation relations $[E_1, E_2] = E_3, [E_2, E_3] = E_1, [E_3, E_1] = E_2$

Problem 2.2.14. Find the basis for $\mathfrak{so}(3)$.

2.3 Mixed Topics: Homomorphisms and Complexification

Theorem 2.3.1. *Let G, H be matrix Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\Phi : G \to H$ be a Lie group homomorphism. Then $\exists! \phi : \mathfrak{g} \to \mathfrak{h}$ a real linear map, with $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$. Then ϕ has following properties: (i) $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$ for all $X \in \mathfrak{g}, A \in G$. (ii) $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. (iii) $\phi(X) = \frac{d}{dt}\Phi(e^{tX})|_{t=0}$ for all $X \in \mathfrak{g}$.*

Proof. Since Φ is continuous, $\Phi(e^{tX})$ is then a one-parameter subgroup of H. Then by Theorem 1.6.5, $\exists! Z$ such that $\Phi(e^{tX}) = e^{tZ}$ for $t \in \mathbb{R}$. Define $\phi(X) = Z$, we check ϕ has the properties.

We check the linearity. First, $\Phi(e^X) = e^Z = e^{\phi(X)}$. Then $\Phi(e^{tsX}) = e^{tsZ}$ Thus $\phi(sX) = s\phi(X)$. Then by Lie product formula and the continuity of Φ, $e^{t\phi(X+Y)} = \Phi(\lim_{m\to\infty}(e^{\frac{tX}{m}} e^{\frac{tY}{m}})^m) = \lim_{m\to\infty}(\Phi(e^{\frac{tX}{m}})\Phi(e^{\frac{tY}{m}}))^m = \lim_{m\to\infty}(e^{\frac{t\phi(X)}{m}} e^{\frac{t\phi(Y)}{m}})^m) = e^{t(\phi(X)+\phi(Y))}$. Therefore, differentiating at $t = 0$ we get $\phi(X + Y) = \phi(X) + \phi(Y)$.

Next, we check uniqueness. If both ϕ, ϕ' has the above property, then $e^{t\phi(X)} = e^{t\phi'(X)} = \Phi(e^{tX})$. Differentiate the result at $t = 0$, we have $\phi(X) = \phi'(X)$.

The we verify the rest properties: $A \in G$, then $e^{t\phi(AXA^{-1})} = e^{\phi(tAXA^{-1})} = \Phi(e^{tAXA^{-1}}) = \Phi(A)\Phi(e^{tX})\Phi(A) = \Phi(A)e^{t\phi(X)}\Phi(A)^{-1}$. Then we can differentiate at $t = 0$ to get (i).

As before, $\phi([X,Y]) = \phi(\frac{d}{dt}e^{tX}Ye^{-tX}|_{t=0}) = \frac{d}{dt}\phi(e^{tX}Ye^{-tX})|_{t=0} = \frac{d}{dt}\Phi(e^{tX})\phi(Y)\Phi(e^{-tX})|_{t=0} = \frac{d}{dt}e^{t\phi(X)}\phi(Y)e^{-t\phi(X)}|_{t=0} = [\phi(X),\phi(Y)]$. Thus (ii). Finally, $\Phi(e^{tX} = e^{\phi(tX)} = e^{t\phi(X)}$, then we compute $\phi(X)$ to get (iii). □

Proposition 2.3.2. *Let G, H, K be matrix Lie groups, $\Phi : H \to K$, $\Psi : G \to H$ be Lie group homomorphisms. Let $\Lambda : G \to K$ and $\Lambda = \Phi \circ \Psi$. Let ϕ, ψ, λ be Lie algebra maps associated to Φ, Ψ, Λ. Then $\lambda = \phi \circ \psi$.*

Proof. Let $X \in \mathfrak{g}$, $\Lambda(e^{tX}) = \Phi(\Psi(e^{tX})) = \Phi(e^{t\psi(X)}) = e^{t\phi(\psi(X))}$. Thus $\lambda(X) = \phi(\psi(X))$ by derivating at $t = 0$. □

Bibliography

[1] Brian C. Hall,