Hilbert Functions

Ting Gong
Commutative Algebra Chapter 11

May 15, 2020

In this section, we let $A = \bigoplus_{n=0}^{\infty} A_n$ be a Noetherian graded ring. Then by 10.7, A_0 is a Noetherian ring and A if finitely generated by say $x_1, \ldots, x_s$ which we take to be homogenous of degrees $k_1, \ldots, k_s$, and M be a finitely generated graded A-module.

Definition 0.1. Let λ be an additive function (with values in $\mathbb{Z}$) on the class of all finitely generated A_0-modules. The *Poincaré series* of M with respect to λ is the generating function of $\lambda(M_n)$, $P(M,t) = \sum_{t=0}^{\infty} \lambda(M_n)t^n \in \mathbb{Z}[[t]]$

Theorem 0.2 (Hilbert, Serre)**.** *$P(M,t)$ is a rational function in t of the form $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$, where $f(t) \in \mathbb{Z}[t]$.*

Proof. We do induction on the number of generators of A over A_0. Assume $s = 0$, then $A_n = 0$ for all $n > 0$, so $A = A_0$ and M is a finitely generated A_0-module. Hence $M_n = 0$ for large n. Thus $P(M,t)$ is a polynomial.

Assume $s > 0$ and the theorem works for $s - 1$. Multiplication by x_s gives an A-module homomorphism of $M_n \to M_{n+k_s}$. Then we have an exact sequence $0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0$. Let $K = \bigoplus K_n$, $L = \bigoplus L_n$, since K is a submodule and L is a quotient module of M, they are both finitely-generated A-modules, and both are annihilated by x_s. Thus are $A_0[x_1, \ldots, x_{s-1}]$-modules. Then we apply λ, by Proposition 2.11 $\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$. Now we multiply by t^{n+k_s} and we sum with respect to n. We have

$$\sum_n t^{k_s}(t^n\lambda(K_n) - t^n\lambda(M_n)) + \sum_n (t^{n+k_s}\lambda(-M_{n+k_s}) + t^{n+k_s}\lambda(L_{n+k_s})) = 0$$
$$t^{k_s}P(K,t) - t^{k_s}P(M,t) + P(M,t) - P(L,t) - g(t) = 0$$

Thus $P(M,t) = (P(L,t) - t^{k_s}P(K,t))/(1-t^{k_s})$, then we use the induction hypothesis to get our desired result. □

The order of the pole of $P(M,t)$ at $t = 1$ is denoted as $d(M)$.

Corollary 0.3. *If each $k_i = 1$, then for all sufficient large n, $\lambda(M_n)$ is a polynomial in n of degree $d - 1$. (This we call the* Hilbert function (polynomial) *of M).*

Proof. By the theorem, $\lambda(M_n)$ is the coefficient of t^n of $f(t)/(1-t)^{-s}$. Cancelling powers, we assume $s=d$ and $f(1)\neq 0$. Assume $f(t)=\sum_{t=0}^N a_k t^k$ since $(1-t)^{-d}=\sum_{k=0}^{\infty}\begin{pmatrix} d+k-1 \\ d-1 \end{pmatrix} t^k$, we have $\lambda(M_n)=\sum_{k=0}^N a_k \begin{pmatrix} d+n-k-1 \\ d-1 \end{pmatrix}$ for $n\geq N$. And the sum on the right is a polynomial in n with leading term $(\sum a_k)n^{d-1}/(d-1)!\neq 0$. □

Proposition 0.4. *If $x\in A_k$ is not a zero divisor in M, then $d(M/xM)=d(M)-1$.*

Proof. Consider the exact sequence in the proof of 0.2, replacing x_s by $x\in A_k$, then $K=0$ and by the computation in 0.2, $d(L)=d(M)-1$. □

Example 0.5. Let $A=A_0[x_1,\dots,x_s]$, where A_0 is an Artin ring. Then A_n is a free A_0-module generated by $x_1^{m_1}\dots x_s^{m_s}$, and $P(A,t)=(1-t)^{-s}$.

Proposition 0.6. *Let A be Noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\mathfrak{q}$ $\mathfrak{m}$-primary ideal, M finitely generated A-module, (M_n) a stable $\mathfrak{q}$-filtration of M. Then:*

(i) M/M_n is of finite length, for each $n\geq 0$

(ii)For all sufficient large n this length is a polynomial $g(n)$ of degree $\leq s$, where s is the least number of generators of $\mathfrak{q}$.

(iii) The degree and leading coefficent of $g(n)$ depend only on M and $\mathfrak{q}$ not on filtration.

Proof. (i) Let $G(A)=\bigoplus \mathfrak{q}^n/q^{n+1}$, $G(M)=\bigoplus M_n/M_{n+1}$. $G_0(A)=A/\mathfrak{q}$ is an Artin local ring by 8.5, and $G(A)$ is Noetherian, $G(M)$ is a finitely generated graded $G(A)$-module by 10.22. Each $G_n(M)$ is a Noetherian A-module annihilated by $\mathfrak{q}$, hence a Noetherian $A/\mathfrak{q}$-module; thus has finite length. Thus M/M_n has finite length, $l_n=l(M/M_n)=\sum_{r=1}^n l(M_r/M_{r+1})$

(ii) If $x_1,\dots,x_s$ generate $\mathfrak{q}$, the images $\bar{x}_i$ in $\mathfrak{q}/\mathfrak{q}^2$ generate $G(A)$ as an $A/\mathfrak{q}$-algebra, and each $\bar{x}_i$ has degree 1. Then by Cor 0.3, $l(M_n/M_{n+1})=f(n)$, a polynomial of degree $\leq s-1$ for large n. And from 0.2, $l_{n+1}-l_n=f(n)$, l_n is a polynomial $g(n)$ with degree $\leq s$.

(iii) Let (M'_n) be another stable $\mathfrak{q}$-filtration of M, let $g'(n)=l(M/M'_n)$. By 10.6, the two filtration has bounded difference, then $\exists n_0$ such that $g(n+n_0)\geq g'(n), g'(n+n_0)\geq g(n)$. Since g,g' are polynomials for large n, $\lim_{n\to\infty} g(n)/g'(n)=1$. Thus they have the same degree and leading coefficient. □

We let $\chi_{\mathfrak{q}}^M(n)=l(M/\mathfrak{q}^n M)$ for large n. If $M=A$, we write $\chi_{\mathfrak{q}}(n)$ and call it the charateristic polynomial of $\mathfrak{q}$.

Corollary 0.7. *For large n, $l(A/\mathfrak{q}^n)$ is $\chi_{\mathfrak{q}}(n)$ of degree $\leq s$, where s is the least number of generators of $\mathfrak{q}$*

Proposition 0.8. $\deg \chi_{\mathfrak{q}}(n)=\deg \chi_{\mathfrak{m}}(n)$

Proof. By 7.16, $\mathfrak{m}\supset\mathfrak{q}\supset\mathfrak{m}^r$ for some r. Then $\mathfrak{m}^n\supset\mathfrak{q}^n\supset\mathfrak{m}^{rn}$. Thus $\chi_{\mathfrak{m}}(n)\leq\chi_{\mathfrak{q}}(n)\leq\chi_{\mathfrak{m}}(rn)$ for large n. Let $n\to\infty$, with χ being polynomials, we have $\deg\chi_{\mathfrak{q}}(n)=\deg\chi_{\mathfrak{m}}(n)$. □

The degree of $\chi_{\mathfrak{q}}(n)$ will be $d(A)$, then by 0.3 $d(A)=d(G_{\mathfrak{m}}(A))$.