Topologies and Completions

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Commutative Algebra Chapter 10

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Definition 0.1. A *topological group* G is a group which is also a topological space, where multiplication $\cdot : G \times G \to G$, $\cdot : (g, h) \mapsto gh$, and inverse map $\iota : G \to G$, $\iota : g \mapsto g^{-1}$ are both continuous.

In this section, we are talking about G being abelian (with addition as our operation, where the two continuous operations are $+ : (g, h) \mapsto g + h, \iota : g \mapsto -g$), and not necessarily Hausdorff.

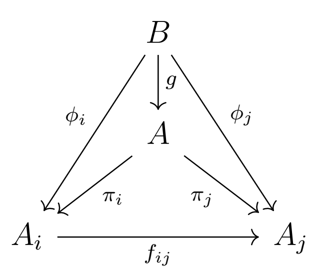
If $\{0\}$ is closed, then the diagonal is closed in $G \times G$, and so G is Hausdorff. If a is fixed in G, the translation $T_a(x) = x + a$ is a homeomorphism of G onto G. Hence if U is a neighborhood of 0 in G, then $U + a$ is a neighborhood of a in G. Conversely every a appears in this form. Thus the topology of G is uniquely determined by the neighborhoods of 0 in G.

Lemma 0.2. *Let H be the intersection of all neighborhoods of 0 in G. Then (i) H is a subgroup, (ii) H is the closure of $\{0\}$, (iii) G/H is Hausdorff, (iv) G is Hausdorff iff $H = 0$.*

The proof is not hard. Can read book if you wish.

Definition 0.3. Let $\{A_i\}$ be a family of groups, $\{f_{ij}\}$ be a set of homomorphisms where $f_{ij} : A_j \to A_i$, $i \leq j$, satisfying the following: (i) f_{ii} is the identity map on A_i, (ii) $f_{ik} = f_{ij} \circ f_{jk}$. Then the pair (A_i, f_{ij}) is called a *inverse system*. If f_{ij} are surjective, then we say it is a surjective system

Definition 0.4. Let (A_i, f_{ij}) be an inverse system, A is the *inverse limit* if there exists a homomorphism $\pi_i : A \to A_i$ satisfying $\pi_i = f_{ij} \circ \pi_j$ for $i \leq j$, and if there is another pair (B, ϕ_i) satisfying the above properties, then there exists a unique homomorphism $g : B \to A$, such that the following diagram commutes, with $\phi_i = \pi_i \circ g$. We denote $A = \varprojlim A_i$.



Remark 0.5. For groups, it is $\varprojlim A = \{a \in \prod A_i | a_i = f_{ij}(a_j) \text{ where } i \leq j\}$

Proposition 0.6. *If $0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ is an exact sequence of inverse systems then $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$ is exact. If $\{A_n\}$ is a surjective system then $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$ is exact.*

Proof. Let $A = \prod_{n=1}^{\infty} A_n$ and $d^A : A \to A$ by $d^A(a_n) = a_n - \theta_{n+1}(a_{n+1})$ (where $\theta_{n+1} : A_{n+1} \to A_n$). Then $\ker d^A \cong \varprojlim A_n$. Define B, C and d^B, d^C accordingly. The exact sequence of inverse systems then defines a commutative diagram of exact sequence as in the book (I don't want to draw this on Latex). Hence by 2.10, we have an exact sequence $0 \to \ker d^A \to \ker d^B \to \ker d^C \to \text{coker} d^A \to \text{coker} d^B \to \text{coker} d^C \to 0$. We then solve $x_n - \theta_{n+1}(x_{n+1}) = a_n$ for $x_n \in A_n$, with $a_n \in A_n$, then we see that d^A is surjective if A_n is surjective. Thus we are good when replacing terms in the exact sequence. □

Definition 0.7. Let $\hat{G}$ be the group of all equivalence classes of the Cauchy sequences in G. And $\hat{G}$ is the *completion* of G. If $G \cong \hat{G}$, then G is *complete*

Corollary 0.8. *Let $0 \to G' \to G \xrightarrow{p} G'' \to 0$ be an exact sequence of group. Let G has the topology defined by a sequence $\{G_n\}$ of subgroups, and G', G'' induced topologies. Then $0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$ is exact.*

Corollary 0.9. *$\hat{G}_n$ is a subgroup of $\hat{G}$ and $\hat{G}/\hat{G}_n \cong G/G_n$*

Proposition 0.10. $\hat{\hat{G}} \cong \hat{G}$

Example 0.11. Let $G = A, G_n = \mathfrak{a}$, where $\mathfrak{a}$ is an ideal in A. The topology on A is called the $\mathfrak{a}-$adic topology. Since $\mathfrak{a}^n$ are ideals, A is a topological ring. By 10.1, the topology is Hausdorff iff $\cap \mathfrak{a}^n = \{0\}$. The completion $\hat{A}$ of A is a topological ring, $\phi : A \to \hat{A}$ is continuous ring homomorphism whose kernel is $\cap \mathfrak{a}^n$. (Likewise we can do it for A-modules.

For A-module M, take $G = M, G_n = \mathfrak{a}^n M$. Thus defines the $\mathfrak{a}$-adic topology on M and $\hat{M}$, the completion of M is a topological $\hat{A}$-module since $\hat{A} \times \hat{M} \to \hat{M}$ is continuous. If $f : M \to N$ is an A-module homomorphism, then $f(\mathfrak{a}^n M) = \mathfrak{a}^n f(M) \subseteq \mathfrak{a}^n N$; thus f is continuous and so defines $\hat{f} : \hat{M} \to \hat{N}$

(i) $A = k[x]$, $\mathfrak{a} = (x)$, then $\hat{A} = k[[x]]$, the formal power series.

(ii) $A = \mathbb{Z}, \mathfrak{a} = (p)$, then $\hat{A}$ is the ring of *p-adic integers*, whose elemenst are infinite series $\sum_{n=0}^{\infty} a_n p^n$, $0 \leq a_n \leq p - 1$.

Definition 0.12. An infinite chain $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n \supseteq \ldots$ where M_n are submodules of M is called a *filtration* of M denoted by (M_n). It is an $\mathfrak{a}$-*filtration* if $\mathfrak{a} M_n \subseteq M_{n+1}$ for all n and a *stable* $\mathfrak{a}$-*filtration* if $\mathfrak{a} M_n = M_{n+1}$ for all sufficiently large n. (Thus $(\mathfrak{a}^n M)$ is a stable $\mathfrak{a}$-filtration).

Lemma 0.13. *If $(M_n), (M_n')$ are stable $\mathfrak{a}$-filtrations of M, then they have bounded difference (i.e. $\exists n_0$ such that $M_{n+n_0} \subseteq M_n'$ and $M_{n+n_0}' \subseteq M_n$ for all $n \geq 0$). Hence all stable $\mathfrak{a}$-flirtations determine the same topology on M, namely the $\mathfrak{a}$-topology.*

Proof. Take $M_n' = \mathfrak{a}^n M$. Since $\mathfrak{a} M \subseteq M_{n+1}$ for all n, we have $\mathfrak{a}^n M \subseteq M_n$. Also, $\mathfrak{a} M_n = M_{n+1}$ for all $n \geq n_0$ for some n_0. Then $M_{n+n_0} = \mathfrak{a}^n M_{n_0} \subseteq \mathfrak{a}^n M$. □